

On the convergence of linear martingales

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§ 1. Introduction

The following generalization of the notion of martingale has been studied by MACQUEEN [7]. Let $(\xi_t, F_t, t=1, 2, \dots)$ be an adapted sequence of random variables and suppose that

$$(1) \quad E\{\xi_s | F_{s-1}\} = a_1 \xi_{s-1} + \dots + a_m \xi_{s-m}$$

for $s > m$, where m is a fixed positive integer and $a_1, \dots, a_m$ are non-random coefficients. From equation (1) we get that

$$(2) \quad \xi_s = a_1 \xi_{s-1} + \dots + a_m \xi_{s-m} + \delta_s,$$

where δ_s is a martingale difference. This autoregressive scheme is widely studied in the literature. Deep investigations have been devoted in particular to the stationarity of the process ξ_s in the Gaussian case (see [1], p. 108). However, it has been pointed out in [7], that the process ξ_s is more closely related to a martingale than a stationary process, if the coefficients a_k are positive and $\sum_{k=1}^m a_k = 1$. In this special case classical martingale convergence theorems of Doob are true for ξ_s (see [7], Section 3).

The aim of this paper is to give a new proof for the results of MacQueen (part (a) and (b) of our Theorem 3). Part (c) of Theorem 3 is new.

Our method is the following: we reduce our problem with the help of the transformation described in (6) and (7) to the study of an m -dimensional process X_t for which

$$(3) \quad E\{X_t | F_{t-1}\} = \Lambda X_{t-1}.$$

We prove some general results for the vector-valued process X_t (Theorems 1 and 2) which imply the convergence of ξ_t .

§ 2. Definitions and preliminary remarks

Definition 1. Let (Ω, F, P) be a probability space, F_s ($s=1, 2, \dots$) an increasing sequence of σ -subalgebras of F and ξ_s ($s=1, 2, \dots$) real random variables (r.v.'s) defined on (Ω, F, P) .

We call the process $(\xi_s, F_s, s=1, 2, \dots)$ a linear martingale if ξ_s is F_s -meas-

urable, $E|\xi_s| < \infty$ for every s and

$$(4) \quad E\{\xi_s | F_{s-1}\} = a_1(s)\xi_{s-1} + \dots + a_m(s)\xi_{s-m}$$

for all $s > m$, where m is a fixed positive integer, and $a_k(s)$ ($k=1, \dots, m; s > m$) are nonnegative non-random coefficients for which $\sum_{k=1}^m a_k(s) = 1$ ($s > m$). (We suppose that there exists an s for which $a_m(s) \neq 0$.)

From equation (4) it follows that

$$(5) \quad \xi_s = a_1(s)\xi_{s-1} + \dots + a_m(s)\xi_{s-m} + \delta_s,$$

where $\delta_s = \xi_s - E\{\xi_s | F_{s-1}\}$ is a martingale difference.

If the initial r.v.'s $\xi_1, \dots, \xi_m$, the coefficients

$$a_k(s) \quad (k = 1, \dots, m; s = m+1, m+2, \dots)$$

and the martingale difference ($\delta_s, F_s, s = m+1, m+2, \dots$) are given, then there exists a process ($\xi_s, F_s, s = 1, 2, \dots$) satisfying (5) and thus also (4).

In order to study the convergence properties of the process ξ_t we introduce the m -dimensional vectors

$$(6) \quad X_t = \begin{pmatrix} \xi_t \\ \vdots \\ \xi_{t-m+1} \end{pmatrix}, \quad \Delta_t = \begin{pmatrix} \delta_t \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and a matrix of type $m \times m$:

$$(7) \quad A(t) = \begin{pmatrix} a_1(t) & \dots & a_m(t) \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}.$$

With these notations we have

$$(8) \quad E\{X_t | F_{t-1}\} = A(t)X_{t-1}$$

and

$$(9) \quad X_t = A(t)X_{t-1} + \Delta_t$$

for $t > m$.

Remark 1. This transformation is widely used to study discrete time autoregressive processes ([1], p. 109).

For the sake of brevity a process ($X_t, F_t, t = 1, 2, \dots$) with property (8) will be called a A -martingale:

Definition 2. An adapted m -dimensional stochastic process ($X_t, F_t, t = 1, 2, \dots$) is called a A -martingale if X_t is integrable and $E\{X_{t+1} | F_t\} = A(t+1)X_t$ for $t = 1, 2, \dots$, where $A(t)$ is a given non-random matrix ($t = 1, 2, \dots$). (Equivalently, (X_t, F_t) is a A -martingale if (9) is satisfied, where (Δ_t, F_t) is a martingale difference.)

From equation (9) one easily deduces that X_t has the following representation

$$(10) \quad X_t = A(t, s)X_s + \sum_{u=s+1}^t A(t, u)\Delta_u \quad (t > s),$$

where the matrices $A(t, u)$ are the solutions of the equations

$$(11) \quad \begin{aligned} A(t, u) &= A(t)A(t-1, u) \quad \text{for } t > u, \\ A(u, u) &= I \quad (\text{the identity matrix}). \end{aligned}$$

Equations (10) and (11) can be considered as the discrete analogues of the solution of a stochastic differential equation (see Theorem 4.2.4 of [4]).

In the sequel $\|\cdot\|$ denotes the norm of a vector or a matrix.

§ 3. Convergence of A -martingales

We shall prove that under certain conditions the classical martingale convergence theorems of Doob are true for A -martingales. We assume that the limit

$$(12) \quad \lim_{t \rightarrow \infty} A(t, u) = A(u)$$

exists for every $u=1, 2, \dots$.

Let us introduce the accompanying martingale of X_t :

$$Y_t = \sum_{u=1}^t A(u)\Delta_u,$$

where $\Delta_1 = X_1$.

Lemma 1. *If the A -martingale (X_t, F_t) is bounded in L_α ($\alpha \geq 1$), i.e.*

$$\sup_t E\|X_t\|^\alpha \leq c < \infty,$$

then its accompanying martingale (Y_t, F_t) is also bounded in L_α :

$$\sup_t E\|Y_t\|^\alpha \leq c.$$

PROOF. Let us consider the following martingale:

$$Y_{t,s} = \sum_{u=1}^s A(t, u)\Delta_u$$

for $1 \leq s \leq t$, where t is fixed. For the submartingale $\|Y_{t,s}\|^\alpha$ ($1 \leq s \leq t$) we have

$$E\|Y_{t,s}\|^\alpha \leq E\|Y_{t,t}\|^\alpha = E\|X_t\|^\alpha \leq c$$

for every $s \leq t$. Since

$$\lim_{t \rightarrow \infty} Y_{t,s} = \sum_{u=1}^s A(u)\Delta_u = Y_s$$

for every fixed s , we get by Fatou's lemma $E\|Y_s\|^\alpha \leq c$.

In addition to assumption (12), we need the following stability condition for the solution of equation (11):

$$(13) \quad \|A(t, u) - A(u)\| \leq c_{t-u} \quad (t \geq u),$$

where $\sum_{s=0}^{\infty} c_s < \infty$.

In the sequel, we suppose that there exists a positive function $C_1(\omega)$ for which

$$(14) \quad C_1(\omega) \|A_u(\omega)\| \leq \|A(u) A_u(\omega)\|$$

for every $u \geq 1$ and $\omega \in \Omega$.

Theorem 1. Let $(X_t, F_t, t=1, 2, \dots)$ be a A -martingale satisfying conditions (13) and (14).

(a) If $\sup_t E\|X_t\| < c < \infty$, then $\lim_{t \rightarrow \infty} X_t = X_\infty$ almost surely (a.s.) and $E\|X_\infty\| < \infty$.

(b) X_t converges in L_1 as $t \rightarrow \infty$ if and only if the family $\{X_t; t=1, 2, \dots\}$ is uniformly integrable.

PROOF. (a) Let Y_t be the accompanying martingale of X_t . By Lemma 1 $\sup_t E\|Y_t\| < c$. It follows from Doob's theorem (see [5], p. 319) that $\lim_{t \rightarrow \infty} Y_t = Y_\infty$ a.s. and $E\|Y_\infty\| < \infty$. Condition (13) implies that

$$(15) \quad \begin{aligned} \|X_t - Y_t\| &= \left\| \sum_{u=1}^t [A(t, u) - A(u)] A_u \right\| \leq \sum_{u=1}^t c_{t-u} \|A_u\| = \\ &= \sum_{s=0}^{t-1} \|A_{t-s}\| c_s = \sum_{s=0}^n \|A_{t-s}\| c_s + \sum_{s=n+1}^{t-1} \|A_{t-s}\| c_s. \end{aligned}$$

It has been proved that the series $\sum_{u=1}^t A(u) A_u = Y_t$ is convergent a.s. Therefore by (14) $\lim_{u \rightarrow \infty} A_u(\omega) = 0$ and $\sup_u \|A_u(\omega)\| \leq c(\omega) < \infty$ for almost all $\omega \in \Omega$. Now inequality (15) and condition $\sum_{s=0}^{\infty} c_s < \infty$ together imply that $\lim_{t \rightarrow \infty} \|X_t - Y_t\| = 0$ a.s. So, in view of $\lim_{t \rightarrow \infty} Y_t = Y_\infty$ a.s. we have $\lim_{t \rightarrow \infty} X_t = X_\infty = Y_\infty$ a.s.

(b) Uniform integrability implies that $\sup_t E\|X_t\| < c < \infty$. By part (a) of this theorem $\lim_{t \rightarrow \infty} X_t = X_\infty$ a.s. hence X_t converges also in L_1 because of the uniform integrability.

Conversely, convergence in L_1 always implies uniform integrability.

Remark 2. If $A(t, u) \rightarrow A(u)$ as $t \rightarrow \infty$, then the accompanying martingale Y_t has the form $Y_t = A(t) X_t$, $t \geq 1$.

Indeed, by (10) and (11)

$$A(s, t) X_t = \sum_{u=1}^t A(s, t) A(t, u) A_u = \sum_{u=1}^t A(s, u) A_u$$

for $s > t$, where $A_1 = X_1$. If s tends to infinity we get $Y_t = A(t) X_t$.

Remark 3. (a) Under conditions (13) and (14) uniform integrability of X_t implies uniform integrability of Y_t . Indeed, from equation

$$E\{X_{u+t}|F_u\} = A(t, u)X_u$$

for $t \rightarrow \infty$ we get

$$E\{X_\infty|F_u\} = A(u)X_u = Y_u.$$

(b) If in (14) $C_1(\omega) = C_1 = \text{const.}$, then the uniform integrability of Y_t implies the uniform integrability of X_t . Indeed, by the proof of Theorem 1 $\lim_{t \rightarrow \infty} \|X_t - Y_t\| = 0$ in L_1 .

(c) The decomposition $X_t = Y_t + Z_t$ is the (unique) Riesz decomposition of the uniformly integrable A -martingale X_t .

Remark 4. Let B be a Banach space with the Radon—Nikodym property (cf. [2]). Theorem 1 is valid for B -valued A -martingales too.

Theorem 2. Let (X_t, F_t) be a A -martingale. Suppose that $\|A(t, u)\| \leq K$ for $t \geq u$. Assume that conditions (12) and (14) hold and in (14) $C_1(\omega) = C_1 = \text{const.}$ If $\sup_t E\|X_t\|^\alpha < \infty$, where $\alpha > 1$, then $\lim_{t \rightarrow \infty} X_t = X_\infty$ in L_α (and a.s.).

PROOF. It follows from Lemma 1 that $\sup_t E\|Y_t\|^\alpha < \infty$ for the accompanying martingale Y_t . By the theorem of Doob $\lim_{t \rightarrow \infty} Y_t = Y_\infty$ in L_α . From this and from the inequality of Burkholder (cf. [3], p. 384) we infer that

$$\begin{aligned} E\left\|\sum_{u=s+1}^t A(t, u)\Delta_u\right\|^\alpha &\leq B_1 E\left(\sum_{u=s+1}^t \|A(t, u)\Delta_u\|^2\right)^{\alpha/2} \leq \\ &\leq B_1 K^\alpha E\left(\sum_{u=s+1}^t \|\Delta_u\|^2\right)^{\alpha/2} \leq B_1 K^\alpha C_1^{-\alpha} E\left(\sum_{u=s+1}^t \|A(u)\Delta_u\|^2\right)^{\alpha/2} \leq \\ &\leq B_1 K^\alpha C_1^{-\alpha} B_2^{-1} E\left\|\sum_{u=s+1}^t A(u)\Delta_u\right\|^\alpha = B_1 K^\alpha C_1^{-\alpha} B_2^{-1} E\|Y_t - Y_s\|^\alpha < \varepsilon \end{aligned}$$

if $s > s_\varepsilon$ for every $t > s$.

Finally, from equality

$$X_t - Y_\infty = (Y_s - Y_\infty) + \left(\sum_{u=1}^s A(t, u)\Delta_u - Y_s\right) + \sum_{u=s+1}^t A(t, u)\Delta_u$$

we get the desired convergence.

§ 4. Convergence of linear martingales

Let $(\xi_t, F_t, t=1, 2, \dots)$ be a linear martingale. Assume that the coefficients $a_k(s)$ do not depend on s : $a_k(s) = a_k$, $s > m$, $k=1, 2, \dots, m$. Let d be the greatest common divisor of those integers k for which $a_k > 0$.

Theorem 3. Let us suppose that $d=1$.

(a) If $\sup_t E|\xi_t| < \infty$, then ξ_t converges almost surely as $t \rightarrow \infty$.

(b) ξ_t converges in L_1 as $t \rightarrow \infty$ iff the family $\{\xi_t; t=1, 2, \dots\}$ is uniformly integrable.

(c) Let $\alpha > 1$. If $\sup_t E|\xi_t|^\alpha < \infty$, then ξ_t converges in L_α as $t \rightarrow \infty$.

PROOF. In § 2 we have constructed the A -martingale $X(t)$, the matrix $A(t)$ and the martingale difference Δ_t corresponding to ξ_t . By the conditions of our theorem $A=A(t)$ is the transition matrix of a non-decomposable acyclic Markov chain with m states. From the theory of Markov chains it is well known that the elements of the matrices $A(t, u)=A^{t-u}$ converge exponentially fast to the elements of the matrix $A(u)=A=(a_{ij})$ as $t \rightarrow \infty$, where $a_{ij}=p_j$ ($i, j=1, \dots, m$) and $p_j = \sum_{i=1}^m a_i / \sum_{i=1}^m ia_i$ ($j=1, \dots, m$) is the unique stationary distribution of the Markov chain.

Therefore conditions (13) and (14) are true. Thus Theorems 1 and 2 imply the present result.

Remark 5. The first component of the martingale Y_t :

$$\eta_t = \left(\sum_{k=1}^m \left(\sum_{i=k}^m a_i \right) \xi_{t+1-k} \right) / \sum_{i=1}^m ia_i$$

can be regarded as an accompanying martingale of ξ_t .

Remark 6. (a) In the case of $d > 1$ Theorem 3 implies the convergence of the subsequences $\xi_i, \xi_{i+d}, \xi_{i+2d}, \dots$ for every $1 \leq i < d$.

(b) Let (ξ_t, F_t) and (η_t, G_t) be independent martingales and suppose that $\lim_{t \rightarrow \infty} \xi_t \neq \lim_{t \rightarrow \infty} \eta_t$. The linear martingale $\xi_1, \eta_1, \xi_2, \eta_2, \dots$ is not convergent (case $d > 1$).

Acknowledgement. The author wishes to thank Professor YU. A. ROZANOV for his support during the preparation of this paper and Professor C. C. HEYDE for references [6] and [7].

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(Received March 10, 1985)