

Remarks on Hyers's theorem

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Abstract. In this note we study the connections between Hyers's theorem on the stability of linear functional equations and amenability. We prove a stronger version of Hyers's theorem on amenable semigroups and state some open problems concerning Hyers's theorem.

In this note C denotes the set of complex numbers and throughout the paper G is a fixed semigroup with identity e . We shall use the operators of left and right translations ${}_yT$ and T_y defined for any function $f: G \rightarrow C$ by

$${}_yTf(x) = f(yx), \quad T_yf(x) = f(xy)$$

further, the left and right difference operators ${}_y\Delta$ and Δ_y defined by

$${}_y\Delta = {}_yT - I, \quad \Delta_y = T_y - I$$

where I denotes the identity operator. For the products ${}_{y_1}\Delta \dots {}_{y_n}\Delta$ and $\Delta_{y_n} \dots \Delta_{y_1}$ we use the notations ${}_{y_1, \dots, y_n}^n\Delta$ and $\Delta_{y_1, \dots, y_n}^n$, respectively further ${}^0\Delta = \Delta^0 = I$. For any integer $n \geq 0$ we let

$$B_n(G) = \{f: G \rightarrow C \mid (x, y_1, \dots, y_{n+1}) \rightarrow \Delta_{y_1, \dots, y_{n+1}}^{n+1}f(x) \text{ is bounded}\},$$

$$P_n(G) = \{f: G \rightarrow C \mid \Delta_{y_1, \dots, y_{n+1}}^{n+1}f(x) = 0 \text{ for all } x, y_1, \dots, y_{n+1} \text{ in } G\},$$

$$M_n(G) = \{f: G \rightarrow C \mid \Delta_{y, \dots, y}^nf(x) = n!f(y) \text{ for all } x, y \text{ in } G\}.$$

Obviously we have $M_n(G) \subseteq P_n(G) \subseteq B_n(G)$ for all n . The elements of $P_n(G)$ are called polynomials of degree at most n and the elements of $M_n(G)$ are called monomials of degree n . The elements of $B_0(G)$ are just the bounded complex functions on G , further $P_0(G)$ and $M_0(G)$ can be identified with C . We note, that $B_n(G)$ and $P_n(G)$ are also meaningful for $n = -1$, and $B_{-1}(G) = B_0(G)$, $P_{-1}(G) = \{0\}$.

We introduce a seminorm on $B_n(G)$ for $n \geq 0$ by

$$\|f\|_n = \sup_{y_1, \dots, y_{n+1}} |\Delta_{y_1, \dots, y_{n+1}}^{n+1}f(e) + (-1)^nf(e)|.$$

If $n = -1$ we let $\|f\|_{-1} = \|f\|_0$. It is easy to see that $\|f\|_n = 0$ if and only if f is a polynomial of degree at most n with $f(e) = 0$. Obviously, $\|f\|_0$ is just the sup-norm of f in $B_0(G)$.

Now we formulate a property of G for all n , which relates to Hyers's theorem. We say that G has the property (H_n) if there exists a mapping $F_n: B_n(G) \rightarrow P_n(G)$ such that

- (H1) F_n is linear,
 (H2) $F_n(p) = p$ for all p in $P_n(G)$ with $p(e) = 0$,
 (H3) $I - F_n$ maps $B_n(G)$ into $B_0(G)$ continuously.

This definition is motivated by the fact that Hyers's theorem can be formulated as follows:

Theorem 1. (HYERS, [5].) *If G is a commutative semigroup with identity, then it has property (H_n) for all n .*

(See also [1], [6].) At the same time, not all semigroups with identity have property (H_n) . Forti [3] gave an example, which shows that the free group on two generators fails to have property (H_1) . Nevertheless, there are lots of noncommutative semigroups having the property (H_n) for all n . To see this we need the notion of amenable semigroups. We say that G is amenable, if there exists a nonnegative linear functional $M: B_0(G) \rightarrow \mathbb{C}$ which is translation invariant and normalized in the sense $M(1) = 1$. Nonnegativity means that $M(f) \geq 0$ for all $f \geq 0$, and translation invariance means, that $M(yTf) = M(T_y f) = M(f)$ for all y in G and f in $B_0(G)$. Such a functional M is called an invariant mean on G . It is known (see e.g. [4]), that all commutative semigroups amenable, but the free group on two generators fails to be amenable. The problem of characterizing all amenable groups or semigroups is still unsolved (see [4]).

Returning to Hyers's theorem, we proved in [7], that if G is amenable, then it has property (H_n) for all n . It means, we have

Theorem 2. (SZÉKELYHIDI, [7].) *If G is an amenable semigroup with identity, then it has property (H_n) for all n .*

Actually, we can prove more, but first we formulate another property of G , which seems to be stronger than (H_n) and may enlighten the relation between amenability and the validity of Hyers's theorem. We say, that G has property (P_n) , if there exists a mapping $\Phi_n: B_n(G) \rightarrow M_n(G)$ such that

- (P1) Φ_n is linear,
 (P2) $\Phi_n(m) = m$ for all m in $M_n(G)$,
 (P3) $I - \Phi_n$ maps $B_n(G)$ into $B_{n-1}(G)$ continuously,
 (P4) Φ_n is translation invariant.

The aim of this work is to prove the following statements:

1. If (P_k) holds for $k = 1, \dots, n$ then (H_k) holds for $k = 1, \dots, n$.
2. (P_0) is equivalent to amenability.
3. (P_0) implies (P_n) for all n .

Theorem 3. *If G has property (P_k) for $k = 1, \dots, n$, then it has property (H_k) for $k = 1, \dots, n$.*

PROOF. Let f be arbitrary in $B_k(G)$, then by (P_k) we have

$$f = \Phi_k(f) + f_1$$

where f_1 belongs to $B_{k-1}(G)$, and $\|f_1\|_{k-1} \leq C_1 \|f\|_k$. Now, by (P_{k-1}) we have

$$f_1 = \Phi_{k-1}(f_1) + f_2$$

with some f_2 in $B_{k-2}(G)$, and $\|f_2\|_{k-2} \leq C_2 \|f\|_k$. Continuing this process we obtain

$$f = \Phi_k(f) + \Phi_{k-1}(f_1) + \dots + \Phi_1(f_{k-1}) + f_k,$$

where f_k is in $B_0(G)$ and $\|f_k\|_0 \leq C_k \|f\|_k$. If we define

$$F_k(f) = \Phi_k(f) + \Phi_{k-1}(f_1) + \dots + \Phi_1(f_{k-1})$$

then we see that F_k maps $B_k(G)$ into $P_k(G)$ linearly, and $I - F_k$ maps $B_k(G)$ into $B_0(G)$ continuously. Hence we have (H1) and (H3). To prove (H2), first we show that $\Phi_k(p) = 0$ for all p in $P_{k-1}(G)$. By the results of [7], each polynomial has a unique representation as the sum of monomials. Hence, it is enough to prove that $\Phi_k(m) = 0$ for all monomials m of degree smaller than k . Of course, we may suppose, that $k \geq 1$. First we show that $\Phi_k(1) = 0$. Let a be any additive function (that is, any element of $M_1(G)$), then we have

$$\Phi_k(a) = \Phi_k(T_y a) = \Phi_k(a + a(y)) = \Phi_k(a) + a(y) \Phi_k(1).$$

It follows, that $\Phi_k(1) \neq 0$ implies $a = 0$ for all a in $M_1(G)$. But this means, that $M_n(G) = \{0\}$ for all $n \geq 1$, and in particular $\Phi_k(1) = 0$. Now suppose that $\Phi_k(m) = 0$ for all monomials m of degree smaller than $k-1$, and let m be arbitrary in $M_{k-1}(G)$ ($k \geq 2$). Let $a \neq 0$ be any element of $M_1(G)$ (we may suppose by the above considerations, that there exists such an a). Then $m \cdot a$ belongs to $M_k(G)$ and we have

$$\Phi_k(m \cdot a) = \Phi_k(T_y(m \cdot a)) = \Phi_k(T_y m \cdot a) + a(y) \cdot \Phi_k(T_y m).$$

It follows from [7], but it can also be proved directly, that

$$T_y m = m + p_y + q_y,$$

where p_y is in $M_{k-2}(G)$ and q_y is in $P_{k-3}(G)$. Hence $q_y \cdot a$ is in $P_{k-2}(G)$, and by our assumption we have

$$\Phi_k(m \cdot a) = \Phi_k(m \cdot a) + \Phi_k(p_y \cdot a) + a(y) \cdot \Phi_k(m),$$

that is

$$\Phi_k(p_y \cdot a) + a(y) \cdot \Phi_k(m) = 0,$$

for all y in G . If $y = x$, then $\Phi_k(m) \neq 0$ implies that $a \cdot \Phi_k(m)$ is of degree $k+1$, and this contradicts the fact, that Φ_k maps into $M_k(G)$. This contradiction proves our statement. Now let p be arbitrary in $P_k(G)$ with $p(e) = 0$. We express p as a sum of monomials

$$p = m_k + m_{k-1} + \dots + m_1$$

with m_j in $M_j(G)$ ($j = 1, \dots, k$). Hence, with the above notations we have

$$\Phi_k(p) = m_k$$

and

$$\Phi_k(p_j) = m_j$$

where $p_j = p_{j+1} - \Phi_{j+1}(p_{j+1})$ for $j = 1, \dots, k-1$, and $p_k = p$. It follows, that

$$F_k(p) = \Phi_k(p) + \Phi_{k-1}(p_{k-1}) + \dots + \Phi_1(p_1) = m_k + \dots + m_1 = p,$$

which was to be proved.

Theorem 4. *G is amenable if and only if it has property (P_0) .*

PROOF. Let G be amenable, and let M be any invariant mean on G . If we let $\Phi_0 = M$, then obviously Φ_0 maps $B_0(G)$ into $M_0(G) = C$, further Φ_0 is linear and translation invariant. Condition (P2) is trivially fulfilled, and we evidently have (P3), as for all f in $B_0(G)$

$$\|f - \Phi_0(f)\|_{-1} = \|f - M(f)\|_0 \leq \|f\|_0 + |M(f)| \leq 2\|f\|_0$$

as the norm of the normalized linear functional M is obviously 1. Conversely, suppose that G has property (P_0) . It means, that there exists a nonzero, translation invariant bounded linear functional Φ_0 on $B_0(G)$. Then obviously the total variation $|\Phi_0|$ (see e.g. [2]) of Φ_0 is a nonzero, nonnegative translation invariant linear functional on $B_0(G)$. As $|\Phi_0|$ is nonzero and nonnegative, we have $|\Phi_0|(1) \neq 0$, and then $(|\Phi_0|(1))^{-1} \cdot \Phi_0$ is an invariant mean on G .

Theorem 5. *If G has property (P_0) , then it has property (P_n) for all n .*

PROOF. By theorem 4, G is amenable. Let M be any invariant mean on G . Then for all f in $B_n(G)$ the function $k(f)$ defined by

$$k(f)(x) = (-1)^n M_{y_1} \dots M_{y_n} (\Delta_{y_1, \dots, y_n, x}^{n+1} f(e))$$

is bounded, and $f - k(f)$ belongs to $P_n(G)$ (see [7]). Obviously $k(f)(e) = 0$, hence we have

$$f = m_n + m_{n-1} + \dots + m_1 + k(f) + f(e),$$

where m_k belongs to $M_k(G)$. It is easy to see (and it follows from [7]) that

$$m_n(x) = \frac{1}{n!} \Delta_{x, \dots, x}^n (f - k(f))(e)$$

holds for all x in G . Now let

$$\Phi_n(f) = m_n.$$

Obviously Φ_n is linear and $\Phi_n(f)$ is in $M_n(G)$. On the other hand,

$$f - \Phi_n(f) = m_{n-1} + \dots + m_1 + k(f) + f(e),$$

which is an element of $B_{n-1}(G)$. If m is in $M_n(G)$, then

$$\Delta_{y_1, \dots, y_n, x}^{n+1} m(e) = 0$$

for all $y_1, \dots, y_n, x$ in G , hence $k(m) = 0$ and

$$\Phi_n(m) = m.$$

It means, that (P1) and (P2) are fulfilled. We show, that Φ_n is translation invariant. Indeed, for all x, y in G we have

$$\begin{aligned} \Phi_n(T_y f)(x) &= \frac{1}{n!} \Delta_{x, \dots, x}^n (T_y f)(e) + \frac{1}{n!} (-1)^{n+1} M_{y_1} \dots M_{y_n} (\Delta_{x, \dots, x}^n \Delta_{y, \dots, y_n}^n (T_y f)(e)) = \\ &= \frac{1}{n!} \Delta_{x, \dots, x}^n f(y) + \frac{1}{n!} (-1)^{n+1} M_{y_1} \dots M_{y_n} (\Delta_{x, \dots, x}^n \Delta_{y_1, \dots, y_n}^n f(y)) = \Phi_n(f)(x). \end{aligned}$$

Similarly, we obtain $\Phi_n({}_y T f)(x) = \Phi_n(f)(x)$.

Finally, to prove continuity in (P3), we notice that

$$f = \Phi_n(f) + p_{n-1} + k(f) + f(e),$$

where p_{n-1} belongs to $B_{n-1}(G)$, and $p(e)=0$. Further

$$|k(f)(x) + f(e)| = |(-1)^n M_{y_1} \dots M_{y_n} (\Delta_{y_1, \dots, y_n, x}^{n+1} f(e) + (-1)^n f(e))| \leq \|f\|_n$$

for all x in G , that is $\|k(f) + f(e)\|_0 \leq \|f\|_n$. On the other hand, we obviously have

$$\|k(f) + f(e)\|_{n-1} \leq C_n \|k(f) + f(e)\|_0,$$

where C_n depends only on n . Now it follows

$$\begin{aligned} \|f - \Phi_n(f)\|_{n-1} &= \|p_{n-1} + k(f) + f(e)\|_{n-1} = \|k(f) + f(e)\|_{n-1} \leq \\ &\leq C_n \|k(f) + f(e)\|_0 \leq C_n \|f\|_n, \end{aligned}$$

and the theorem is proved.

Corollary 6. *Amenability of G implies that G has the property (P_n) , and hence the property (H_n) for all n .*

Remarks 7. In theorem 3 we were unable to prove that (P_n) is stronger than (H_n) for a fixed n . It would be also interesting to know, whether the converse of theorem 5. is true. If so, at least for $n=1$, then amenability would equivalent to (P_1) , which is a stronger form of Hyers's original theorem. We can state these questions as open problems:

1. Is it true, that (P_n) implies (H_n) ?
2. Is it true, that (P_1) implies amenability?
3. Is it true, that (H_1) implies amenability?

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