

About eigensolutions of abstract differential equations with mixed conditions

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Introduction

Let's take Operational Calculus $CO(L^0, L^1, S, Tq, sq, q, Q)$, where L^0, L^1 are linear spaces, $S: L^1 \rightarrow L^0$ (onto) is a linear operation called derivative.

Linear operations $T(q): L^0 \rightarrow L^1$, $s(q): L^1 \rightarrow \text{Ker } S$, such that $ST(q) = \text{id}_{L^0}$, $T(q)S = \text{id}_{L^1} - s(q)$ are called integral and limit condition. Index $q \in Q$ defines uniquely integral and limit condition (definition and properties of the Operational Calculus see [3]).

We assume, that $\text{Ker } S$ is algebra with unit.

Linear space $\Xi(X)$ is a set of results that is fractions of the form $\frac{f}{u}$, where f is an element of the linear space X , while $U: X \rightarrow X$ is an endomorphism and at the same time an injection belonging to a commutative subgroup $\pi(X)$ (see [3]).

If $A: X \rightarrow X$ is a commutative endomorphism with injections $U \in \pi(X)$, then operation $\mu = \frac{A}{U}$ definition in the results space $\Xi(X)$ with the formula

$$(1) \quad \mu \frac{f}{V} = \frac{Af}{UV}$$

is called an operator.

In particular operator $p = \frac{\text{id}_{L^0}}{Tq}$ is the so called operator of HEAVISIDE (see [3]).

In this work we will use the formula

$$(2) \quad S^n x = p^n x - p^n x_0 - p^{n-1} x_1 - \dots - p x_{n-1},$$

$$x_i = s(q) S^i x, \quad i = 0, 1, \dots, n-1,$$

if only $x \in L^n$ (proof see [3]).

In the space $L^k \subset L^0$, $k \in N$ we introduce the relation of equivalence $\frac{\|k\|}{\bar{q}}$:

$$(3) \quad x \frac{k}{\bar{q}} y \Leftrightarrow \bigwedge_{0 \leq i \leq k-1} s(q) S^i (x-y) = 0 \quad (\text{see [4], [7]}).$$

Set $\mathcal{C}_q^k := L^k / \frac{k}{q}$, whose elements are abstract's classes

$$(4) \quad [x]_q^k = \left[\sum_{i=0}^{k-1} T^i(q) s(q) S^i x \right]_q^k$$

is called a set of Taylor's jets (see [4], [7]). Operation $t^i(q)$ with the property

$$t^i(q)c = [i!T^i(q)c]_q^k, \quad c \in \text{Ker } S$$

is called an abstract variable. Hence

$$\xi := [x]_q^k = \sum_{i=0}^{k-1} \frac{t^i(q)}{i!} x_i, \quad x_i = s(q)S^i x \in \text{Ker } S.$$

CHAPTER I.

EXPOTENTIAL FUNCTIONS. HEAVISIDE'S THEOREMS*

§ 1. Logarithm. Expotential functions and expotential polynomials.

Df. 1.1. Commutative endomorphism R with derivative S and with limit condition $s(q)$ is called a logarithm, if

$$(1.1.1) \quad (I - T(q)R)f = 0 \text{ entails } f = 0 \text{ for } f \in L^0, \text{ or what gives the same result}$$

$$(1.1.2) \quad Sf = Rf, \quad s(q)f = 0 \text{ implies } f = 0.$$

Df. 1.2. If there exist elements $f \neq 0$ satisfying condition $s(q)f = 0$ and equation $Sf = Rf$, then endomorphism $R \neq 0$ is called eigenendomorphism and element f eigenelement. When $R = \varrho \text{id}_{L^0}$, when number ϱ is called eigenvalue.

Theorem 1.1. *If there exists a solution of equation*

$$(1.1.3) \quad Sx = Rx, \quad s(q)x = c, \quad c \in \text{Ker } S, \text{ } R\text{-logarithm,}$$

then there exists only one.

PROOF. Let's assume, that $x_1, x_2, x_1 \neq x_2$ are solutions of the equation (1.1.3). So

$$Sx_1 = Rx_1, \quad s(q)x_1 = c$$

$$Sx_2 = Rx_2, \quad s(q)x_2 = c.$$

Subtracting these equations and applying linear character of logarithm R and then using (1.1.2) we obtain contraction with our assumption, what was to be proved.

Let R be logarithm and let a solution of equation (1.1.3) exist. We define this solution

$$(1.1.4) \quad x = e^{Rt(q)} c$$

* This chapter is written on the basis of [3] § 6.

Df. 1.3. Function

$$(1.1.5) \quad e^{Rtq} : \text{Ker } SL^1$$

given by the formula

$$(1.1.6) \quad c = e^{Rtq} c$$

is called exponential function.

Theorem 1.2. *If result c) $I - TqR$ is an element of space L^0 , then exponential function can be defined with operator $I(I - TqR = P) \quad P - R$ in the following way:*

$$(1.1.7) \quad e^{Rtq} c = \frac{c}{I - TqR} = \frac{P}{p - R} c.$$

PROOF. Let $x = e^{Rtq} c$. Multiplying both sides of the equation (1.1.3) by integral Tq and applying limit condition, we get

$$x - c = TqRx.$$

Dividing by $I - TqR$ is possible because R is logarithm. Hence we have

$$(1.1.8) \quad x = \frac{c}{I - TqR}.$$

Multiplying numerator and denominator of the fraction by p we get

$$(1.1.9) \quad x = \frac{P}{p - R} c,$$

what was to be proved.

Theorem 1.3. *Let's assume, that together with the exponential function*

$$ce^{Rtq} c = \frac{c}{I - TqR}$$

as a result belonging to space L^0 , further results

$$(1.1.10) \quad \frac{T^m(q) c}{(I - T(q)R)^{m+1}} = \frac{P}{(p - R)^{m+1}} c, \quad m = 1, 2, \dots$$

are also elements of space L^0 . Than all these elements belong to space L^∞ . Defining

$$(1.1.11) \quad \frac{t^m(q)}{m!} e^{Rt(q)} c := \frac{T^m(q) c}{(I - T(q)R)^{m+1}},$$

we have also

$$(1.1.12) \quad \begin{aligned} S \left(\frac{t^m(q)}{m!} e^{Rt(q)} c \right) &= \frac{t^{m-1}(q)}{(m-1)!} e^{Rt(q)} c + \frac{t^m(q)}{m!} e^{Rt(q)} Rc, \\ s \left(\frac{t^m(q)}{m!} e^{Rt(q)} c \right) &= 0 \quad \text{for } m = 1, 2, \dots \end{aligned}$$

PROOF. Let's notice the following identity:

$$(1.1.13) \quad \frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}} = \frac{T^{m-1}(q)c}{(I-T(q)R)^m} + \frac{T^m(q)Rc}{(I-T(q)R)^{m+1}}.$$

From the assumption it follows, that the right side is an element of space L^0 . In such a case

$$T(q) \frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}} \in L^1,$$

i.e.

$$\frac{T^m(q)c}{(I-T(q)R)^{m+1}} \in L^1$$

for every natural m . Through induction we get that all these elements are the elements of space L^∞ .

Of course

$$(1.1.14) \quad S\left(T(q) \frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}\right) = \frac{T^{m-1}(q)c}{(I-T(q)R)^{m-1}}$$

and

$$(1.1.15) \quad s\left(T(q) \frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}\right) = 0,$$

so we have proved that element

$$T(q) \frac{T^{m-1}(q)c}{(I-T(q)R)^{m+1}}$$

is an element of space L^1 . Applying definition 1.1.11 we get

$$\frac{T^m(q)c}{(I-T(q)R)^{m+1}} = \frac{t^m(q)}{m!} e^{Rt(q)} c,$$

$$\frac{T^{m-1}(q)c}{(I-T(q)R)^m} = \frac{t^{m-1}(q)}{(m-1)!} e^{Rt(q)} c,$$

$$\frac{T^m(q)Rc}{(I-T(q)R)^{m+1}} = \frac{t^m(q)}{m!} e^{Rt(q)} Rc.$$

From identity (1.1.13) and formulas (1.1.14), (1.1.15) we obtain formulas (1.1.12). We get (1.1.10) multiplying the numerator and denominator of the left side of the identity by p^{m+1} .

Results $\frac{t^m(q)}{m!} e^{Rt(q)} c$ are called expotential functions if they belong to space L^0 .

Df. 1.4. Linear combination of arbitrary expotential functions is called expotential polynomial.

It is then of the form

$$(1.1.16) \quad W(t(q)) = \sum_{j=1}^m \sum_{k_j=0}^n \frac{t^{k_j}(q)}{k_j!} e^{R_j t(q)} c_{jk_j}.$$

On the basis of (1.1.10) expotential polynomial has also the form

$$(1.1.17) \quad V(p) = \sum_{j=1}^m \sum_{k_j=0}^{n_j} \frac{p}{(p-R_j)^{k_j+1}} c_{jk_j} \in \Xi(L^0).$$

§ 2. Heaviside's theorem about expotential functions of the operator p .

Theorem 2.1. (The first Heaviside's theorem). *In case when the degree of the numerator does not exceed the degree of the denominator and when polynomial $H(p)$ can be presented in the form*

$$(1.2.1) \quad H(p) = B_m(p-R_1)(p-R_2) \cdot \dots \cdot (p-R_m),$$

the rational function of operator p

$$(1.2.2) \quad F(p)c := \frac{K(p)}{H(p)} c = \frac{A_n p^n + A_{n-1} p^{n-1} + \dots + A_1 p + A_0}{B_m p^m + B_{m-1} p^{m-1} + \dots + B_1 p + B_0} c,$$

can be defined by the ordinary expotential functions with the formula

$$(1.2.3) \quad F(p)c = \frac{K(0)}{H(0)} c + \sum_{k=1}^m \frac{K(R_k)}{B_m R_k H'(R_k)} e^{R_k t(q)} c,$$

where $A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_m \in \pi(X)$, $R_1, R_2, \dots, R_m$ are logarithms differents from zero, $H'(R_k) = (R_k - R_1) \cdot \dots \cdot (R_k - R_{k-1})(R_k - R_{k+1}) \cdot \dots \cdot (R_k - R_m)$.

PROOF. From the assumption we have

$$\frac{K(p)}{pH(p)} = \frac{K(p)}{B_m p(p-R_1) \cdot \dots \cdot (p-R_m)} = \frac{K_0}{p} + \frac{K_1}{p-R_1} + \dots + \frac{K_m}{p-R_m},$$

because in this case the degree of the numerator is lower from the degree of denominator and there exists distribution into ordinary proper fractions. Hence we have

$$K(p) = B_m K_0(p-R_1) \cdot \dots \cdot (p-R_m) + B_m K_1 p(p-R_2) \cdot \dots \cdot (p-R_m) + \dots + \\ + B_m K_m p(p-R_1) \cdot \dots \cdot (p-R_{m-1})$$

and hence

$$K_k = \frac{K(R_k)}{B_m R_k (R_k - R_1) \cdot \dots \cdot (R_k - R_{k-1})(R_k - R_{k+1}) \cdot \dots \cdot (R_k - R_m)} = \frac{K(R_k)}{B_m R_k H'(R_k)} \\ \text{for } k = 1, 2, \dots, m.$$

All these calculations are permissible, if it is allowed to devide by R_k and $R_i - R_j$;
We have also

$$K(0) = B_m K_0(-R_1) \cdot \dots \cdot (-R_m) = K_0 H(0).$$

hence

$$K_0 = \frac{K(0)}{H(0)}.$$

So

$$F(c) = \frac{K(p)}{H(p)} c = \left(K_0 + K_1 \frac{p}{p-R_1} + \dots + K_m \frac{p}{p-R_m} \right) c = K_0 c + \sum_{j=1}^m K_j e^{R_j t(q)} c,$$

what was to be proved.

Theorem 2.2. (The second Heaviside's theorem). *Let's admit for denominator $H(p)$ a canonical distribution with multiple roots and with zero root.*

$$(1.2.4) \quad H(p) = B_m p^{\alpha_0} (p-R_1)^{\alpha_1} \dots (p-R_l)^{\alpha_l}, \quad \sum_{j=0}^l \alpha_j = m.$$

Then

$$(1.2.5) \quad F(p)c = \frac{K(p)}{H(p)} c = \sum_{r=0}^{\alpha_0} A_{0r} \frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} c + \sum_{k=1}^l \sum_{r=1}^{\alpha_k} A_{kr} \frac{t_{(q)}^{\alpha_k+1-r}}{(\alpha_k+1-r)!} e^{R_k t(q)} c,$$

where

$$(1.2.6) \quad A_{0r} = \left|_{p=0} \frac{1}{r!} \frac{d^r}{dp^r} \left[\frac{p^{\alpha_0} K(p)}{H(p)} \right], \right.$$

$$(1.2.7) \quad A_{kr} = \left|_{p=R_k} \frac{1}{(r-1)!} \frac{d^{r-1}}{dp^{r-1}} \left[\frac{(p-R_k)^{\alpha_k} K(p)}{pH(p)} \right]. \right.$$

PROOF. Similarly as in the proof of theorem 2.1 we look for distribution of function $\frac{F(p)}{p} c$ into common fractions, obtaining

$$\begin{aligned} \frac{K(p)}{pH(p)} c &= \frac{A_{00}}{p^{\alpha_0+1}} c + \frac{A_{01}}{p^{\alpha_0}} c + \dots + \frac{A_{0\alpha_0}}{p} c + \frac{A_{11}}{(p-R_1)^{\alpha_1}} c + \frac{A_{12}}{(p-R_1)^{\alpha_1-1}} c + \dots + \\ &+ \frac{A_{1\alpha_1}}{(p-R_1)^1} c + \dots + \frac{A_{l1}}{(p-R_l)^{\alpha_l}} c + \dots + \frac{A_{l\alpha_l}}{(p-R_l)^1} c. \end{aligned}$$

Multiplying the last identity by p^{α_0+1} and then derivating r -fold with regard to variable p we obtain

$$\begin{aligned} \frac{d^r}{dp^r} \left[\frac{p^{\alpha_0} K(p)}{H(p)} \right] &= r! A_{0r} + (r+1)! p A_{0,r+1} + \dots + \frac{\alpha_0!}{(\alpha_0-r)!} p^{\alpha_0-r} A_{0,\alpha_0} + \\ &+ (\alpha_0+1) p^{\alpha_0} \left[\frac{A_{11}}{(p-R_1)^{\alpha_1}} + \dots + \frac{A_{l\alpha_l}}{(p-R_l)^1} \right] + p^{\alpha_0+1} \frac{d^r}{dp^r} \left[\frac{A_{11}}{(p-R_1)^{\alpha_1}} + \dots + \frac{A_{l\alpha_l}}{(p-R_l)^1} \right]. \end{aligned}$$

Substituting $p=0$ we get formula (1.2.6). Similarly, multiplying the considered identity by $(p-R_k)^{\alpha_k}$, then derivating it r -fold and substituting $p=R_k$ we get identity (1.2.7).

Remembering, that

$$\frac{1}{p^k} c = \frac{t^k(q)}{k!} c, \quad \frac{p}{(p-R_k)^{j_k+1}} c = \frac{t^{j_k}(q)}{j_k!} e^{R_k t(q)} c$$

we get (1.2.5).

We will use formula (1.2.5) under the assumption that it is allowed to divide by endomorphisms R_i and $R_j - R_k$.

CHAPTER II.
EIGENSOLUTIONS OF THE ABSTRACT DIFFERENTIAL LINEAR
EQUATIONS OF THE ORDER N AND THE SYSTEM OF
EQUATIONS OF THE FIRST ORDER

§ 1. The existence of the eigensolutions of the abstract differential linear equation with mixed conditions.

Let's consider linear differential equations of the order n

$$(2.1.1) \quad a_n S^n x + a_{n-1} S^{n-1} x + \dots + a_1 S x + a_0 x = \lambda x, \\ x \in L^n, \quad \lambda, a_i \in \pi(L^1).$$

Onto solution x we impose conditions on the mixed type

$$(2.1.2) \quad \sum_{j=1}^n \sum_{\beta=0}^{r_j} a_{\beta j}^{(\gamma)} x_{\beta j} = 0, \quad \gamma = 1, 2, \dots, w, a_{\beta j}^{(\gamma)} \in \text{Ker } S,$$

where

$$(2.1.3) \quad x_{\beta j} := \sigma(t_j) S^\beta x = |_{t(q)=t_j} ((S^\beta x), \quad \beta = 0, 1, \dots, r_j; j = 1, \dots, N,$$

while

$$(2.1.4) \quad t_j \frac{m}{\tau_0} t_0^{(j)} + \frac{\tau(\tau_0)}{1!} t_1^{(j)} + \dots + \frac{\tau^{m-1}(\tau_0)}{(m-1)!} t_{m-1}^{(j)}, \quad t_i^{(j)} \in \text{Ker } S_\tau, \quad i = 0, 1, \dots, m-1,$$

is Taylor's jet of the class $\mathcal{C}_{\tau_0}^m$, where S_τ is arbitrary derivative (see [4], [7]).

Applying formulas (2) and (2.1.1) we get

$$(2.1.6) \quad a_n (p^n x - p^n x_0^0 - p^{n-1} x_1^0 - \dots - p x_{n-1}^0) + a_{n-1} (p^{n-1} x - p^{n-1} x_0^0 - \\ - p^{n-2} x_1^0 - \dots - p x_{n-2}^0) + \dots + a_1 (p x - p x_0^0) + a_0 x = \lambda x.$$

Hence

$$(2.1.7) \quad x = \sum_{i=0}^{n-1} \frac{W_i(p)}{W(p, \lambda)} x_i^0, \quad x_i^0 = s(q) S^i x, \quad i = 0, 1, \dots, n-1,$$

where

$$(2.1.8) \quad W(p, \lambda) = a_n p^n + a_{n-1} p^{n-1} + \dots + a_1 p + (a_0 - \lambda)$$

$$(2.1.9) \quad W_i(p) = a_n p^{n-i} + a_{n-1} p^{n-i+1} + \dots + a_{i+1} p \quad \text{for } i = 0, 1, \dots, n-1.$$

Formula (2.1.7) can be applied when the roots of polynomial $W(p, \lambda)$ are logarithms of derivative S .

Let $0, R_1, R_2, \dots, R_l$ be the roots of the equation $W(p, \lambda)=0$ with multiplication factors respectively $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_l$. Applying the Second Heaviside's Theorem we get

$$(2.1.10) \quad x = \sum_{i=0}^{n-1} \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i)} \frac{t_j^{\alpha_0+1-r}}{(\alpha_0+1-r)!} + \sum_{k=1}^l \sum_{r=1}^{\alpha_k} A_{kr}^{(i)} \frac{t_j^{\alpha_k+1-r}}{(\alpha_k+1-r)!} e^{R_k t(q)} \right] x_i^0,$$

where

$$(2.1.11) \quad A_{0r}^{(i)} = \left| \frac{1}{r!} \frac{d^r}{dp^r} \left[\frac{p^{\alpha_0} W_i(p)}{W(p, \lambda)} \right] \right|_{p=0}$$

$$A_{kr}^{(i)} = \left| \frac{1}{(r-1)!} \frac{d^{r-1}}{dp^{r-1}} \left[\frac{(p-R_k)^{\alpha_k} W_i(p)}{pW(p, \lambda)} \right] \right|_{p=R_k}.$$

It is easy to notice that

$$\sigma(t_j) S^\beta \frac{t_j^{\alpha_0+1-r}}{(\alpha_0+1-r)!} = \begin{cases} \frac{t_j^{\alpha_0+1-r-\beta}}{(\alpha_0+1-r-\beta)!} & \text{for } \alpha_0+1-r-\beta \geq 0 \\ 0 & \text{for } \alpha_0+1-r-\beta < 0 \end{cases},$$

similarly for

$$\frac{t_j^{\alpha_k+1-r}}{(\alpha_k+1-r)!} e^{R_k t(q)}.$$

Acting with operation σ onto general solution (2.1.10) we have

$$(2.1.12) \quad x_{\beta j} = \sum_{i=0}^{n-1} \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i)} \frac{t_j^{\alpha_0+1-r-\beta}}{(\alpha_0+1-r-\beta)!} + \sum_{k=1}^l \sum_{r=1}^{\alpha_k} A_{kr}^{(i)} \sigma(t_j) \left\{ \sum_{v=0}^{\beta} \binom{\beta}{v} \times \right. \right. \\ \left. \left. \times \left(\frac{t_j^{\alpha_k+1-r-v}}{(\alpha_k+1-r-v)!} e^{R_k t(q)} + \frac{t_j^{\alpha_k+1-r}}{(\alpha_k+1-r)!} R_k^{\beta-v} e^{R_k t(q)} \right) \right\} \right] x_i^0.$$

Substituting the so defined $x_{\beta j}$ into equations (2.1.2) we get the system

$$(2.1.13) \quad \sum_{j=1}^N \sum_{\beta=0}^{r_j} a_{\beta j}^{(j)} \sum_{i=0}^{n-1} \left[A_{0r}^{(i)} \frac{t_j^{\alpha_0+1-r-\beta}}{(\alpha_0+1-r-\beta)!} + \sum_{k=1}^l \sum_{r=1}^{\alpha_k} A_{kr}^{(i)} \sigma(t_j) \left\{ \sum_{v=0}^{\beta} \binom{\beta}{v} \times \right. \right. \\ \left. \left. \times \left(\frac{t_j^{\alpha_k+1-r-v}}{(\alpha_k+1-r-v)!} e^{R_k t(q)} + \frac{t_j^{\alpha_k+1-r}}{(\alpha_k+1-r)!} R_k^{\beta-v} e^{R_k t(q)} \right) \right\} \right] x_i^0 = 0,$$

System (2.1.13) is a system of homogeneous W equations with unknown $x_0^0, x_1^0, \dots, x_{n-1}^0$.

If the order of the matrix a of this system is smaller than n , then equation (2.1.1) with conditions (2.1.12) has eigensolutions defined with the formula (2.1.10), where constants $x_0^0, x_1^0, \dots, x_{n-1}^0$ are solutions of the system (2.1.3), where the order of the matrix is equals r (then we write $r(a)=r$), if there exists a reversible determinant of the order r formed from the elements of this matrix while all the determinants of the order higher than r are not reversible (see [8] corollary on page 359). Let's con-

sider in particular the case where polynomial $W(p, \lambda)$ has non-zero roots with multiplication one: $R_1, R_2, \dots, R_n$, while condition (2.1.2) is of the form

$$(2.1.14) \quad \sum_{\beta=0}^{r_1} A_{\beta}^{(\gamma)} x_{\beta 1} = \sum_{\beta=0}^{r_2} B_{\beta}^{(\gamma)} x_{\beta 2}; \quad x_{\beta j} = |_{t(q)=t_j} (S^{\beta} x), \quad j = 1, 2; \quad \gamma = 1, 2, \dots, W.$$

Applying to the formula the first Heaviside's theorem we get

$$(2.1.15) \quad x = \sum_{i=0}^{n-1} [b_i + \sum_{k=1}^n d_{ik} e^{R_k t(q)}] x_i^0,$$

where

$$b_i := \frac{W_i(0)}{W(0, \lambda)}, \quad d_{ik} := \frac{W_i(R_k)}{a_k R_k W'(R_k)}.$$

Because from the assumption we have that $W(0, \lambda)$ is reversible (we write $W(0, \lambda) \in \text{Inv}$), and for every $i=0, 1, \dots, n-1$, $W_i(0)=0$, hence $b_i=0$.

Taking denotation

$$(2.1.16) \quad c_k := \sum_{i=0}^{n-1} d_{ik} x_i^0$$

finally we have formula

$$(2.1.17) \quad x = \sum_{k=1}^n e^{R_k t(q)} c_k.$$

Substituting the obtained general solution to (2.1.14) we get system of homogeneous equations

$$(2.1.18) \quad \sum_{\beta=0}^{\max(r_1, r_2)} \sum_{k=1}^n R_k^{\beta} (A_{\beta}^{(\gamma)} e^{R_k t_1} - B_{\beta}^{(\gamma)} e^{R_k t_2}) c_k = 0, \quad r = 1, 2, \dots, W,$$

with unknowns $c_1, c_2, \dots, c_n$. Condition

$$(2.1.19) \quad r(\alpha) < n,$$

where

$$\alpha = [a_{ij}]_{\substack{1 \leq i \leq W \\ 1 \leq j \leq n}}, \quad a_{ij} = \sum_{\beta=0}^{\max(r_1, r_2)} R_j^{\beta} (A_{\beta}^{(i)} e^{R_j t_1} - B_{\beta}^{(i)} e^{R_j t_2})$$

is necessary and sufficient for equation (2.1.1) with conditions (2.1.14) to have eigensolutions defined with the formula (2.1.17) where constants satisfy system (2.1.18).

Example 1. Let $L^0 = L^1 = L^2 = C(N)$ be a space of sequences of real numbers.

Derivative $S: C(N) \rightarrow C(N)$, integral $T(k_0)$ and limit condition $s(k_0)$ are defined with the formulas

$$\begin{aligned} S\{x_k\} &= \{x_{k+1} - x_k\} \\ T(k_0)\{x_k\} &= \begin{cases} 0 & \text{for } k = k_0 \\ x_{k_0} + x_{k_0+1} + \dots + x_{k-1} & \text{for } k_0 < k \\ -x_{k_0-1} - x_{k_0-2} - \dots - x_k & \text{for } k_0 > k \end{cases} \\ s(k_0)\{x_k\} &= \{x_{k_0}\} \quad (\text{see [5]}). \end{aligned}$$

Let's consider equation

$$(*) \quad S^2 \{x_k\} = \lambda \{x_k\}$$

with conditions

$$(**) \quad \begin{cases} x_{k_1} - x_{k_2+1} = 0 \\ 3x_{k_1} + 3x_{k_2} - x_{k_1+1} - x_{k_2+1} = 0 \end{cases}$$

which are conditions of the type (2.1.14) because they can be written in the form

$$(***) \quad \begin{cases} \sigma(k_1) \{x_k\} - \sigma(k_2) \{x_k\} - \sigma(k_2) S \{x_k\} = 0 \\ 2\sigma(k_1) \{x_k\} - \sigma(k_1) S \{x_k\} + 2\sigma(k_2) \{x_k\} - \sigma(k_2) S \{x_k\} = 0 \end{cases}$$

where operation $\sigma(k_i)$ means substitution of natural number k_i instead of variable k .

Polynomial $W(p, \lambda) = p^2 - \lambda$ has non-zero roots $R_1 = \sqrt{\lambda}$ and $R_2 = -\sqrt{\lambda}$. Because in this model

$$e^{Rr(q)} c = (1 + R)^k c$$

hence applying (2.1.17) we get general solution of the equation (*)

$$x_k = (1 + \sqrt{\lambda})^k c_1 + (1 - \sqrt{\lambda})^k c_2,$$

and hence

$$\sigma(k_i) \{x_k\} = (1 + \sqrt{\lambda})^{k_i} c_1 + (1 - \sqrt{\lambda})^{k_i} c_2$$

$$\sigma(k_i) S \{x_k\} = [(1 + \sqrt{\lambda})^{k_i} c_1 - (1 - \sqrt{\lambda})^{k_i} c_2] \sqrt{\lambda}, \quad i = 1, 2.$$

Substituting previous formulas to conditions (***) we get the following system of homogenous equations with unknown c_1 and c_2 :

$$[(1 + \sqrt{\lambda})^{k_1} - (1 + \sqrt{\lambda})^{k_2+1}] c_1 + [(1 - \sqrt{\lambda})^{k_1} - (1 - \sqrt{\lambda})^{k_2+1}] c_2 = 0$$

$$(2 - \sqrt{\lambda}) [(1 + \sqrt{\lambda})^{k_1} + (1 + \sqrt{\lambda})^{k_2}] c_1 + (2 + \sqrt{\lambda}) [(1 - \sqrt{\lambda})^{k_1} + (1 - \sqrt{\lambda})^{k_2}] c_2 = 0.$$

Thys systems has non-zero solutions if

$$\begin{pmatrix} *** \\ * \end{pmatrix} \begin{vmatrix} (1 + \sqrt{\lambda})^{k_1} - (1 + \sqrt{\lambda})^{k_2+1} & (1 - \sqrt{\lambda})^{k_1} - (1 - \sqrt{\lambda})^{k_2+1} \\ (2 - \sqrt{\lambda}) [(1 + \sqrt{\lambda})^{k_1} + (1 + \sqrt{\lambda})^{k_2}] & (2 + \sqrt{\lambda}) [(1 - \sqrt{\lambda})^{k_1} + (1 - \sqrt{\lambda})^{k_2}] \end{vmatrix} = 0$$

Conditions $\begin{pmatrix} *** \\ * \end{pmatrix}$ defines how eith given λ we must take values k_1 and k_2 so that equation (*) with conditions (**) should have eigensolutions.

And so for example if $\lambda = 4$, then we get connection

$$(3^{k_1} - 3^{k_2+1}) [(-1)^{k_1} + (-1)^{k_2}] = 0,$$

and hence we have

$$k_2 = k_1 - 1, \quad k_1 = 1, 2, \dots, \quad \forall \quad k_2 = k_1 + 2l + 1$$

In this case the eigensolutions of the equation (*) with conditions

$$\begin{aligned}\sigma(k_1) \{x_k\} - \sigma(k_1 - 1) \{x_k\} - \sigma(k_1 - 1) S \{x_k\} &= 0 \\ 2\sigma(k_1) \{x_k\} - \sigma(k_1) \{x_k\} + 2\sigma(k_1 - 1) \{x_k\} - \sigma(k_1 - 1) S \{x_k\} &= 0\end{aligned}$$

are elements

$$\{x_k\} = \{3^k \alpha\}, \quad \alpha \in R.$$

§ 2. Existence of eigensolutions of a system of differential equations of the first order with mixed conditions.

Let's consider a system of differential equations of the form

(2.2.1)

$$Sx_1 + a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = \lambda_1 x_1$$

$$Sx_2 + a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = \lambda_2 x_2$$

$$Sx_n + a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = \lambda_n x_n,$$

$$x_i \in L^1, \quad i = 1, 2, \dots, n,$$

where coefficients of this system and $\lambda_1, \lambda_2, \dots, \lambda_n$ are endomorphisms commutative with each other.

To obtain eigensolutions on the system (2.2.1) we will impose conditions of the mixed type

$$(2.2.2) \quad \sum_{k=1}^N \sum_{p=1}^n \sum_{q=1}^{q_{pk}} \alpha_{Wq}^{(p,k)} \sigma(t_k) S^{q-1} x_p = 0, \quad W = 1, 2, \dots, m,$$

where operation $\sigma(t_k)$ is defined by formula (2.1.3). As

$$(2.2.3) \quad Sx_i = px_i - px_{i0},$$

where

$$(2.2.4) \quad x_{i0} = s(q) x_i \in \text{Ker } S,$$

hence from the system (2.2.1) we get

(2.2.5)

$$(p + a_{11} - \lambda_1) x_1 + a_{12} x_2 + \dots + a_{1n} x_n = px_{10}$$

$$a_{21} x_1 + (p + a_{22} - \lambda_2) x_2 + \dots + a_{2n} x_n = px_{20}$$

$$a_{n1} x_1 + a_{n2} x_2 + \dots + (p + a_{nn} - \lambda_n) x_n = px_{n0}.$$

Solution of this system are elements

$$(2.2.6) \quad x_j = \frac{\Delta_j(p, \Lambda)}{\Delta(p, \Lambda)}, \quad j = 1, 2, \dots, n,$$

where

$$(2.2.7) \quad \Delta(p, \Lambda) := \begin{vmatrix} p + a_{11} - \lambda_1 & a_{12} \dots a_{1n} \\ a_{21} & p + a_{22} - \lambda_2 \dots a_{2n} \\ a_{n1} & a_{n2} \dots p + a_{nn} - \lambda_n \end{vmatrix}.$$

$$(2.2.8) \quad \Delta_j(p, \Lambda) = \begin{vmatrix} p + a_{11} - \lambda_1 & a_{12} \dots a_{1j-1} & p x_{10} & a_{1,j+1} \dots a_{1n} \\ a_{21} & p + a_{22} - \lambda_2 \dots a_{2,j-1} & p x_{20} & a_{2,j+1} \dots a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} \dots a_{n,j-1} & p x_{n0} & a_{n,j+1} \dots p + a_{nn} - \lambda_n \end{vmatrix}$$

We can apply formula (2.2.6) if the roots of the equation $\Delta(p, \Lambda) = 0$ $R_1, R_2, \dots, R_1$, are logarithms.

Applying to $\Delta_j(p, \Lambda)$ theorem of Laplace we obtain

$$(2.2.9) \quad x_j = \sum_{i=1}^n \frac{p \Delta_{ij}(p, \Lambda)}{\Delta(p, \Lambda)} x_{i0}, \quad j = 1, 2, \dots, n,$$

where Δ_{ij} is a respective *co-factor*.

Elements x_j defined by the formula (2.2.9) are of the form (2.1.7).

Applying formula (2.1.10) we have

$$(2.2.10) \quad x_j = \sum_{i=1}^n \left[\sum_{r=0}^{\alpha_0} A_{0r}^{(i,j)} \frac{t_{(q)}^{\alpha_0+1-r}}{(\alpha_0+1-r)!} + \sum_{s=1}^i \sum_{r=1}^{\alpha_s} A_{sr}^{(i,j)} \frac{t_{(q)}^{\alpha_s+1-r}}{(\alpha_s+1-r)!} e^{R_s t(q)} \right] x_{i0},$$

where

$$A_{0r}^{(i,j)} = \left|_{p=0} \frac{1}{r!} \frac{d^r}{dp^r} \left[\frac{p^{\alpha_0+1} \Delta_{ij}(p, \Lambda)}{\Delta(p, \Lambda)} \right] \right|$$

(2.2.11)

$$A_{sr}^{(i,j)} = \left|_{p=R_s} \frac{1}{(r-1)!} \frac{d^{r-1}}{dp^{r-1}} \left[\frac{(p - R_s)^{\alpha_s} \Delta_{ij}(p, \Lambda)}{\Delta(p, \Lambda)} \right] \right|, \quad \sum_{s=0}^i \alpha_s = n.$$

Hence and from (2.2.2) we finally get the system

$$(2.2.12) \quad \sum_{k=1}^N \sum_{j=1}^n \sum_{q=1}^{\alpha_{jk}} \sum_{i=1}^n \left[\sum_{r=0}^{\alpha_0} \alpha_{wq}^{(j,k)} A_{0r}^{(i,j)} \frac{t_k^{\alpha_0-q-r+2}}{(\alpha_0-q-r+2)!} + \right. \\ \left. + \sum_{s=1}^i \sum_{r=1}^{\alpha_s} \alpha_{wq}^{(j,k)} A_{sr}^{(i,j)} \sigma(t_k) S^{q-1} \left(\frac{t_{(q)}^{\alpha_s+1-r}}{(\alpha_s+1-r)!} e^{R_s t(q)} \right) \right] x_{i0} = 0, \quad w = 1, 2, \dots, \omega.$$

It is a system of homogeneous ω equations with n unknowns $x_{10}, x_{20}, \dots, x_{n0}$ and if only order of the matrix of this system is smaller than n then the system of equations (2.2.1) with conditions (2.2.2) has eigensolutions (2.2.10), where constants $x_{10}, x_{20}, \dots, x_{n0}$ are solutions of the system (2.2.12).

EXAMPLE 2. Let

$$S = \frac{d}{dt}: C^1(R) \rightarrow C^0(R)$$

$$T(q)f = \int_{t_0}^t f(\tau) d\tau, \quad f \in C^0(R), \quad t_0 \in (a, b)$$

$$s(q)x(t) = x(t_0) = x_0.$$

Let's consider system of equations

$$\begin{aligned} (*) \quad & x_1'(t) - x_1(t) - x_2(t) = \lambda_1 x_1(t) \\ & x_2'(t) + 5x_1(t) + x_2(t) = \lambda_2 x_2(t) \end{aligned}$$

with conditions

$$\begin{aligned} & 4x_1(t_1) + x_1'(t_1) - x_2'(t_2) = 0 \\ & x_1'(t_1) + 2x_1''(t_1) - x_2'(t_2) - x_2''(t_2) = 0 \\ & x_1(t_1) + x_2(t_1) - 2x_1(t_2) + 2x_2'(t_2) - x_1''(t_2) = 0. \end{aligned}$$

To simplify the calculations we will make analysis in case $\lambda_1 = \lambda_2 = 0$.

Applying (2.2.3) we get the system

$$(p-1)x_1 - x_2 = px_{10}$$

$$5x_1 + (p+1)x_2 = px_{20},$$

for which

$$\Delta(p, 0) = p^2 + 4.$$

Hence we have

$$R_1 = 2i, \quad R_2 = -2i.$$

On the bases of formula (2.2.10) and passing from expotential functions to trigonometric functions finely we get general solution of a given system in the form

$$\begin{aligned} x_1 &= \left\{ \left(\cos 2t + \frac{1}{2} \sin 2t \right) x_{10} + \left(\frac{1}{2} \sin 2t \right) x_{20} \right\} \\ x_2 &= \left\{ \left(-\frac{5}{2} \sin 2t \right) x_{10} + \left(\cos 2t - \frac{1}{2} \sin 2t \right) x_{20} \right\}. \end{aligned}$$

Substituting the obtained solution to $(**)$ we get the system

$$\begin{aligned} & (5 \cos 2t_1 + 5 \cos 2t_2) x_{10} + (2 \sin 2t_1 + \cos 2t_1 + 2 \sin 2t_2 + \cos 2t_1) x_{20} = 0 \\ & -6 \sin 2t_1 - 7 \cos 2t_1 + 5 \cos 2t_2 - 10 \sin 2t_2) x_{10} + \\ & + (\cos 2t_1 - 4 \sin 2t_1 + 5 \cos 2t_2) x_{20} = 0 \\ & (\cos 2t_1 - 2 \sin 2t_1 - 8 \cos 2t_2 + \sin 2t_2) x_{10} + \\ & + (\cos 2t_1 - 3 \sin 2t_2 - 2 \cos 2t_2) x_{20} = 0, \end{aligned}$$

which has non-zero solution if all the determinants of the second order created from the matrix of this system vanish, i.e. if

$$\begin{vmatrix} 5 \cos 2t_1 + 5 \cos 2t_2 & 2 \sin 2t_1 + \cos 2t_1 + 2 \sin 2t_2 + \cos 2t_2 \\ -6 \sin 2t_1 - 7 \cos 2t_1 + 5 \cos 2t_2 - 10 \sin 2t_2 & \cos 2t_1 - 4 \sin 2t_1 + 5 \cos 2t_2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 5 \cos 2t_1 + 5 \cos 2t_2 & 2 \sin 2t_1 + \cos 2t_1 + 2 \sin 2t_2 + \cos 2t_2 \\ \cos 2t_1 - 2 \sin 2t_1 - 8 \cos 2t_2 + \sin 2t_2 & \cos 2t_1 - 3 \sin 2t_2 - 2 \cos 2t_2 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} -6 \sin 2t_1 - 7 \cos 2t_1 + 5 \cos 2t_2 - 10 \sin 2t_2 & \cos 2t_1 - 4 \sin 2t_1 + 5 \cos 2t_2 \\ \cos 2t_1 - 2 \sin 2t_1 - 8 \cos 2t_2 + \sin 2t_2 & \cos 2t_1 - 3 \sin 2t_2 - 2 \cos 2t_2 \end{vmatrix} = 0.$$

Hence after conversions we get equations

$$4 \cos 2(t_2 - t_1) + 3 \sin 2(t_2 - t_1) + 4 = 0$$

$$2 \cos 2(t_2 - t_1) - 18 \sin 2(t_2 - t_1) + 2 = 0$$

$$11 \cos 2(t_2 - t_1) + 5 \sin 2(t_2 - t_1) + 11 = 0,$$

which the common solution is

$$t_2 - t_1 = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}_0.$$

It means that the system (*) with conditions

$$4x_1(t_1) + x_1'(t_1) - x_2'\left(t_1 + \frac{\pi}{2} + k\pi\right) = 0$$

$$x_1'(t_1) + 2x_1''(t_1) - x_2'\left(t_1 + \frac{\pi}{2} + k\pi\right) - x_2''\left(t_1 + \frac{\pi}{2} + k\pi\right) = 0$$

$$x_1(t_1) + x_2(t_1) - 2x_1\left(t_1 + \frac{\pi}{2} + k\pi\right) + 2x_2'\left(t_1 + \frac{\pi}{2} + k\pi\right) - x_1''\left(t_1 + \frac{\pi}{2} + k\pi\right) = 0$$

has eigensolutions for arbitrary $t_1 \in \mathbb{R}$ and $k \in \mathbb{Z}$, which are functions of the form

$$x_1 = \{(\cos 2t - \sin 2t)c\}$$

$$x_2 = \{(-3 \cos 2t - \sin 2t)c\}, \quad c \in \mathbb{R}.$$

References

- [1] W. J. ARNOLD, Równania różniczkowe zwyczajne. *PWN, Warszawa*, 1975. Tłumaczenie z rosyjskiego.
- [2] R. BITTNER, Operational calculus in linear spaces. *Studia Math.* **20** (1961), 1—18.
- [3] R. BITTNER, Rachunek operatorów w przestrzeniach liniowych. *PWN, Warszawa*, 1974.
- [4] R. BITTNER and Z. KOBUS — Taylor's jets. Manifolds of solutions of differential equations in the operational calculus. *Preprint No 35, May 1980. University of Gdańsk.*

- [5] R. BITTNER, E. MIEŁOSZYK, About eigenvalues of differential equations in the operational calculus. *Zeszyty Naukowe Politechniki Gdańskiej. Matematyka XI* Nr. 285 s. 87—99. Gdańsk, 1978 r.
- [6] R. BITTNER and Z. SMENTEK, Taylor's formula for many derivatives. Integrals. *Demonstratio Mathematica* 12. Technical University Warsaw, 1—16.
- [7] Z. KOBUS, H. WYŚOCKI — Pochodna Frecheta odwzorowań kielków Taylora. Odwracalność kielków Taylora. *Zeszyty Naukowe WSMW Gdynia* 1983, 95—113.
- [8] S. LANG — Algebra. *PWN Warszawa* 1973 r.

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(Received February 12, 1984.)