

On generalized pseudo symmetric manifolds

By M. C. CHAKI (Calcutta) and S. P. MONDAL (Calcutta)

Abstract. In this paper it is shown that out of four 1-forms in terms of which a generalized pseudo symmetric manifold $G(PS)_n$ is defined, only two are independent. Further, introducing a new notion of hyper quasi-constant curvature of a Riemannian manifold it is proved that a conformally flat $G(PS)_n$ ($n > 3$) of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.

1. Introduction

The notion of a generalized pseudo symmetric manifold was introduced by one of the authors (CHAKI) in a recent paper [1]. A non-flat Riemannian manifold (M^n, g) ($n > 2$) was called by him a generalized pseudo symmetric manifold if its curvature tensor R satisfies the condition:

$$(1) \quad \begin{aligned} (\nabla_X R)(Y, Z, W) &= 2A(X) R(Y, Z, W) + B(Y) R(X, Z, W) \\ &+ C(Z) R(Y, X, W) + D(W) R(Y, Z, X) + g(R(Y, Z, W), X)P, \\ X, Y, Z, W, P &\in \mathfrak{X}(M), \end{aligned}$$

where A, B, C, D are 1-forms, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and P is defined by

$$(2) \quad g(X, P) = A(X) \quad \forall X.$$

The 1-forms A, B, C, D are called the associated 1-forms of the manifold and an n -dimensional manifold of this kind is denoted by $G(PS)_n$. It may be mentioned in this connection that although the definition of a $G(PS)_n$ is similar to that of a weakly symmetric Riemannian manifold studied by TAMÁSSY and BIHN [2], the defining condition of the latter is

a little weaker than that of a $G(P\mathcal{S})_n$. Let

$$(3) \quad \begin{aligned} g(X, Q) &= B(X), & g(X, T) &= C(X) \\ g(X, U) &= D(X) \quad \forall X. \end{aligned}$$

Then P, Q, T, U are called the basic vector fields corresponding to the associated 1-forms A, B, C, D respectively.

Let L be the symmetric endomorphism of the tangent space corresponding at each point to the Ricci tensor S of type $(0, 2)$. Then

$$(4) \quad g(LX, Y) = S(X, Y) \quad \forall X, Y.$$

We write

$$(5) \quad \begin{aligned} A(LX) &= \bar{A}(X), & B(LX) &= \bar{B}(X), \\ C(LX) &= \bar{C}(X), & D(LX) &= \bar{D}(X). \end{aligned}$$

Then $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are called the auxiliary 1-forms corresponding to the associated 1-forms A, B, C, D respectively.

In this paper it is shown that the defining condition of a $G(P\mathcal{S})_n$ can always be expressed in the following form:

$$(6) \quad \begin{aligned} (\nabla_X R)(Y, Z, W) &= 2A(X) R(Y, Z, W) + B(Y) R(X, Z, W) \\ &+ B(Z) R(Y, X, W) + A(W) R(Y, Z, X) + g(R(Y, Z, W), X)P. \end{aligned}$$

Thus out of the four 1-forms A, B, C, D only two are different, namely A and B and out of the four basic vector fields P, Q, T, U only two, namely P and Q are different. For a conformally flat $G(P\mathcal{S})_n$ ($n > 3$), it is proved that if it is of non-zero constant scalar curvature then the Ricci tensor S has the following form:

$$\begin{aligned} S(X, Y) &= a_1 g(X, Y) + b_1 A(X) B(Y) + b_2 A(X) \bar{B}(Y) \\ &+ b_3 B(X) B(Y) + b_4 B(X) \bar{B}(Y) + b_5 \bar{A}(X) B(Y) + b_6 \bar{B}(X) B(Y) \end{aligned}$$

where $a_1, b_1, b_2, \dots, b_6$ are suitable scalars.

Finally, introducing the notion of hyper quasi-constant curvature in a Riemannian manifold, it is shown that a conformally flat $G(P\mathcal{S})_n$ ($n > 3$) of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.

2. Preliminaries

From (1) we get

$$(2.1) \quad \begin{aligned} (\nabla_X 'R)(Y, Z, W, V) &= 2A(X) 'R(Y, Z, W, V) \\ &+ B(Y) 'R(X, Z, W, V) + C(Z) 'R(Y, X, W, V) \\ &+ D(W) 'R(Y, Z, X, V) + A(V) 'R(Y, Z, W, X), \end{aligned}$$

where

$$(2.2) \quad 'R(Y, Z, W, V) = g(R(Y, Z, W), V).$$

Now, contracting (2.1) over Z and W we get

$$(2.3) \quad \begin{aligned} (\nabla_X S)(Y, V) &= 2A(X) S(Y, V) + B(Y) S(X, V) \\ &+ C(R(X, Y, V)) + D(R(X, V, Y)) + A(V) S(X, Y). \end{aligned}$$

Next, contracting (2.3) over Y and V , by (3)–(5) we obtain

$$(2.4) \quad \begin{aligned} dr(X) &= 2A(X)r + S(X, Q) + S(X, T) + S(X, U) + S(X, P) \\ &= 2A(X)r + \bar{B}(X) + \bar{C}(X) + \bar{D}(X) + \bar{A}(X), \end{aligned}$$

where r is the scalar curvature. These formulas will be used in the sequel.

3. Associated 1-forms of a $G(PS)_n$

In this section it will be shown that the four associated 1-forms A, B, C, D of a $G(PS)_n$ cannot be all different.

Interchanging Y and Z in (2.1) we get

$$(3.1) \quad \begin{aligned} (\nabla_X 'R)(Z, Y, W, V) &= 2A(X) 'R(Z, Y, W, V) \\ &+ B(Z) 'R(X, Y, W, V) + C(Y) 'R(Z, X, W, V) \\ &+ D(W) 'R(Z, Y, X, V) + A(V) 'R(Z, Y, W, X). \end{aligned}$$

Now, adding (2.1) and (3.1) we obtain,

$$\begin{aligned} 0 &= B(Y) 'R(X, Z, W, V) + B(Z) 'R(X, Y, W, V) \\ &+ C(Z) 'R(Y, X, W, V) + C(Y) 'R(Z, X, W, V) \end{aligned}$$

or,

$$(3.2) \quad [B(Y) - C(Y)] 'R(X, Z, W, V) + [B(Z) - C(Z)] 'R(X, Y, W, V) = 0.$$

Now, contracting (3.2) over Y and Z we get

$$(3.3) \quad {}'R(W, V, X, Q) - {}'R(W, V, X, T) = 0.$$

From (3.3) it follows that $Q = T$. Hence

$$(3.4) \quad B(X) = C(X) \quad \forall X.$$

Similarly, interchanging W and V in (2.1) and proceeding as before we get

$$(3.5) \quad D(X) = A(X) \quad \forall X.$$

From (3.4) and (3.5) we see that the associated 1-forms A, B, C, D are not all different, because $A = D$ and $B = C$. In virtue of this we can state the following

Theorem 1. *The defining equation of a $G(PS)_n$ can always be expressed in the following form:*

$$\begin{aligned} (\nabla_X {}'R)(Y, Z, W, V) &= 2A(X) {}'R(Y, Z, W, V) \\ &+ B(Y) {}'R(X, Z, W, V) + B(Z) {}'R(Y, X, W, V) \\ &+ A(W) {}'R(Y, Z, X, V) + A(V) {}'R(Y, Z, W, X). \end{aligned}$$

In virtue of this theorem the formula (2.4) takes the following form:

$$(3.6) \quad dr(X) = 2[A(X)r + \bar{A}(X) + \bar{B}(X)].$$

If a $G(PS)_n$ is of non-zero constant scalar curvature, then from (3.6) it follows that $A(X)r + \bar{A}(X) + \bar{B}(X) = 0$.

This leads to the following

Theorem 2. *If in a $G(PS)_n$ the scalar curvature r is a non-zero constant, then*

$$A(X)r + \bar{A}(X) + \bar{B}(X) = 0 \quad \forall X.$$

4. Conformally flat $G(PS)_n$ ($n \geq 3$)

It is known [(4), p. 41 Theorem 4.1] that in a conformally flat (M^n, g) ($n \geq 3$)

$$\begin{aligned} &(\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) \\ (4.1) \quad &= \frac{1}{2(n-1)}[dr(X)g(Y, Z) - dr(Z)g(X, Y)]. \end{aligned}$$

In virtue of (3.6) and Theorem 1 the formula (2.3) takes the following form:

$$(4.2) \quad (\nabla_X S)(Y, V) = 2A(X) S(Y, V) + B(Y) S(X, V) \\ + B[R(X, Y, V)] + A[R(X, V, Y)] + A(V) S(X, Y).$$

Hence, with the help of the Ricci identity we obtain

$$(4.3) \quad (\nabla_X S)(Y, Z) - (\nabla_Z S)(Y, X) = A(X) S(Y, Z) - A(Z) S(Y, X) \\ - 'R(Z, X, Y, Q) - 2 'R(Z, X, Y, P).$$

Using (4.3) and (3.6) we can express (4.1) as follows:

$$(4.4) \quad (n-1)A(X) S(Y, Z) - (n-1)A(Z) S(X, Y) \\ + (n-1) 'R(X, Z, Y, Q) + 2(n-1) 'R(X, Z, Y, P) \\ = g(Y, Z)(A(X)r + \bar{A}(X) + \bar{B}(X)) - g(Y, X)(A(Z)r + \bar{A}(Z) + \bar{B}(Z)).$$

Putting $Z = Q$ in (4.4) we get

$$(4.5) \quad (n-1)A(X) \bar{B}(Y) - (n-1)A(Q) S(X, Y) \\ + (n-1) 'R(X, Q, Y, Q) + 2(n-1) 'R(X, Q, Y, P) \\ = g(Y, Q)(A(X)r + \bar{A}(X) + \bar{B}(X)) - g(X, Y)(A(Q)r + \bar{A}(Q) + \bar{B}(Q)).$$

If the scalar curvature r is a non-zero constant, then in virtue of Theorem 2 the right-hand side of (4.5) vanishes and (4.5) takes the following form:

$$(n-1)A(X) \bar{B}(Y) - (n-1)A(Q) S(X, Y) \\ + (n-1) 'R(X, Q, Y, Q) + 2(n-1) 'R(X, Q, Y, P) = 0$$

or

$$(4.6) \quad A(Q) S(X, Y) = A(X) \bar{B}(Y) + 'R(X, Q, Y, Q) + 2 'R(X, Q, Y, P).$$

In a conformally flat (M^n, g) ($n \geq 3$) we have

$$(4.7) \quad 'R(X, Y, Z, W) = \frac{1}{n-2} [S(Y, Z) g(X, W) - S(X, Z) g(Y, W) \\ + S(X, W) g(Y, Z) - S(Y, W) g(X, Z)] \\ + \frac{r}{(n-1)(n-2)} [g(X, Z) g(Y, W) - g(Y, Z) g(X, W)].$$

Using this formula we can write (4.7) as follows:

$$\begin{aligned}
 (4.8) \quad & {}'R(X, Q, Y, Q) + 2 {}'R(X, Q, Y, P) \\
 &= \frac{1}{n-2} \{ -S(X, Y)[2A(Q) + B(Q)] - g(X, Y)[2\bar{A}(Q) + \bar{B}(Q)] \\
 &\quad + \bar{B}(X) B(Y) + B(X) \bar{B}(Y) + 2[\bar{A}(X) B(Y) + A(X) \bar{B}(Y)] \} \\
 &+ \frac{r}{(n-1)(n-2)} \{ g(X, Y)[2A(Q) + B(Q)] - B(Y)[2A(X) + B(X)] \}.
 \end{aligned}$$

In virtue of (4.8) we can express (4.6) as follows:

$$\begin{aligned}
 & S(X, Y) \left\{ A(Q) + \frac{1}{n-2} [2A(Q) + B(Q)] \right\} \\
 &= A(X) \bar{B}(Y) - \frac{1}{n-2} g(X, Y) [2\bar{A}(Q) + \bar{B}(Q)] \\
 &\quad + \frac{1}{n-2} [\bar{B}(X) B(Y) + B(X) \bar{B}(Y)] \\
 &\quad + \frac{2}{n-2} [\bar{A}(X) B(Y) + A(X) \bar{B}(Y)] \\
 &\quad + \frac{r}{(n-1)(n-2)} [2A(Q) + B(Q) g(X, Y)] \\
 &\quad - \frac{rB(Y)}{(n-1)(n-2)} [2A(X) + B(X)]
 \end{aligned}$$

or

$$\begin{aligned}
 (4.9) \quad & S(X, Y) = a_1 g(X, Y) + b_1 A(X) B(Y) + b_2 A(X) \bar{B}(Y) \\
 & + b_3 B(X) B(Y) + b_4 B(X) \bar{B}(Y) + b_5 \bar{A}(X) B(Y) + b_6 \bar{B}(X) B(Y),
 \end{aligned}$$

where $a_1, b_1, \dots, b_6$ are scalars in terms of r , $A(Q)$ and $B(Q)$. This leads to the following

Theorem 3. *In a conformally flat $G(PS)_n$ of non-zero constant scalar curvature, the Ricci tensor S has the form (4.9).*

5. Hyper quasi-constant curvature

In this section we shall introduce in a Riemannian manifold the notion of hyper quasi-constant curvature as a generalization of the notion of quasi-constant curvature introduced by CHEN and YANO [5]. According to these authors a Riemannian manifold (M^n, g) ($n > 3$) is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor $'R$ of

type (0,4) satisfies the condition

$$(5.1) \quad \begin{aligned} {}^R(X, Y, Z, W) = & a[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \\ & + b[g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z) \\ & + g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)], \end{aligned}$$

where A is a 1-form and a, b are scalars.

We shall generalize this notion as follows: A Riemannian manifold (M^n, g) ($n > 3$) will be said to be of *hyper quasi-constant curvature* if it is conformally flat and its curvature tensor R of type (0,4) satisfies the condition

$$(5.2) \quad \begin{aligned} {}^R(X, Y, Z, W) = & a[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \\ & + g(X, W)\{bAB\}(Y, Z) - g(Y, W)\{bAB\}(X, Z) \\ & + g(Y, Z)\{bAB\}(X, W) - g(X, Z)\{bAB\}(Y, W), \end{aligned}$$

where $\{bAB\} \equiv b_1AB + b_2A\bar{B} + b_3BB + b_4B\bar{B} + b_5\bar{A}B + b_6\bar{B}B$ and $b_1, b_2, \dots, b_6$ are scalars.

If $B = A$, then the $G(PS)_n$ reduces to a pseudo-symmetric manifold $(PS)_n$ introduced by CHAKI in [3]. In that paper it has been proved that in a conformally flat $(PS)_n$ the auxiliary 1-form $\bar{A}$ is proportional to the associated 1-form A [(3), p. 57 (6.7)]. Taking the factor of proportionality as k the relation (5.2) takes the form:

$$(5.3) \quad \begin{aligned} {}^R(X, Y, Z, W) = & a[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \\ & + b_1[g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z) \\ & + g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \end{aligned}$$

where a and b_1 are appropriate scalars and b_1 is related to k in a simple way. In view of (5.1) it follows from (5.3) that in this case the manifold $(PS)_n$ is of quasi-constant curvature. This is the reason why a conformally flat $G(PS)_n$ ($n > 3$) satisfying the condition (5.2) has been called a manifold of hyper quasi-constant curvature. The word ‘hyper’ has been used because the name hypercomplex number has been given to a quaternion as a generalization of a complex number.

The question now arises whether there exists a conformally flat Riemannian manifold whose curvature tensor R satisfies a condition of the form (5.2). The remaining part of this paper provides an answer to this question.

We consider a $G(PS)_n$ ($n > 3$) which is conformally flat and of non-zero constant scalar curvature. We can therefore use the expression (4.9)

for $S(X, Y)$. Substituting this expression for $S(X, Y)$ in (4.7), it can be expressed, after simplification, in the following form:

$$(5.4) \quad \begin{aligned} R(X, Y, Z, W) = & a_1[g(X, Z)g(Y, W) - g(Y, Z)g(X, W)] \\ & + g(X, W)\{b'AB\}(Y, Z) - g(Y, W)\{b'AB\}(X, Z) \\ & + g(Y, Z)\{b'AB\}(X, W) - g(X, Z)\{b'AB\}(Y, W), \end{aligned}$$

where $\{b'AB\} \equiv b'_1AB + b'_2A\bar{B} + b'_3BB + b'_4B\bar{B} + b'_5\bar{A}B + b'_6\bar{B}B$ and $b'_1, b'_2, \dots, b'_6$ are scalars. Comparing (5.4) and (5.2) we conclude that the manifold under consideration is of hyper quasi-constant curvature. We can therefore state the following theorem which provides an answer to the question raised above:

Theorem 4. *A conformally flat $G(PS)_n$ ($n > 3$) of non-zero constant scalar curvature is a manifold of hyper quasi-constant curvature.*

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M. C. CHAKI
27, SASHI BHUSAN DE STREET
CALCUTTA 700 012
INDIA

S. P. MONDAL
C/O. PROF. M.C. CHAKI
27, SASHI BHUSAN DE STREET
CALCUTTA 700 012
INDIA

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