Integrability conditions of an $F(K, K-2)$-structure satisfying $F^K+F^{K-2}=0$ (K=odd)

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Summary: YANO, HOUH and CHEN [1] have studied the structures defined by a tensor field Φ of the type (1, 1) satisfying $\Phi^4 \pm \Phi^2=0$. GADEA and CORDERO [2] have obtained the integrability conditions of these structures. The generalised $F(K, K-2)$-structure has been defined and studied by GUPTA [3]. The purpose of the present paper is to obtain the integrability conditions of a generalised $F(K, K-2)$ structure satisfying $F^K+F^{K-2}=0$, where F is a non-null tensor field of the type (1, 1) and K is odd. Here we have also obtained the conditions of partial integrability (introducing s_K-partial integrability and t_K-partial integrability) and integrability of the generalised $F(K, K-2)$-structure in terms of its *Nijenhuis tensor* for K odd.

1. The operators s and t for an $F(K, K-2)$-structure

Let M^n be an n-dimensional differentiable manifold of class C^∞ equipped with a (1, 1) tensor field $F(F\neq 0)$ and of class C^∞ satisfying [3]

$$n=2m, \quad F^K+F^{K-2}=0, \quad (2 \operatorname{rank} F-\operatorname{rank} F^{K-1})=\dim M^n, \tag{1.1}$$

when K is odd.

The operators s and t have been defined as follows [3]:

$$s=(-1)^{1/2(K-1)} F^{K-1}, \quad t=I-(-1)^{1/2(K-1)} F^{K-1}, \tag{1.2}$$

I denoting the identity operator. Then we have

Theorem (1.1). *For a tensor field $F(F\neq 0)$ satisfying* (1.1), *the operators s and t defined by* (1.2) *and applied to the tangent space at a point of the manifold are complementary projection operators.*

PROOF. By virtue of (1.1) and (1.2), we have

$$s+t=I; \tag{1.3}$$

$$\begin{cases} s^2=(-1)^{K-1}F^{2K-2}=F^K\cdot F^{K-2} \\ \quad =-F^{K-2}\cdot F^{K-2}=-F^K\cdot F^{K-4}= \\ \quad =(-1)^2F^{K-2}\cdot F^{K-4}=(-1)^2F^K\cdot F^{K-6}= \\ \quad \cdots\cdots\cdots\cdots\cdots \\ \quad \cdots\cdots\cdots\cdots\cdots \\ \quad =(-1)^{1/2(K-1)}F^{K-2}\cdot F^{K-(K-1)}, = \\ \quad =(-1)^{1/2(K-1)}F^{K-1}=s; \end{cases} \tag{1.4}$$

$$t^2 = I+(-1)^{K-1}F^{2K-2}-2(-1)^{1/2(K-1)}F^{K-1} = \tag{1.5}$$
$$= I-(-1)^{1/2(K-1)}F^{K-1} = t;$$

$$st = ts = (-1)^{1/2(K-1)}F^{K-1}-(-1)^{K-1}F^{2K-2} = 0. \tag{1.6}$$

This proves the theorem.

Let S and T be the complementary distributions corresponding to the projection operators s and t respectively. Let the rank of F be constant and be equal to r, then $\dim S=2r-n$ and $\dim T=2n-2r$. Here the dimensions of S and T are both even. Obviously, $n\leqq 2r\leqq 2n$. Such a structure has been called a generalised '$F(K, K-2)$-structure of rank r' and the manifold M^n with this structure an '$F(K, K-2)$-manifold' [3].

Theorem (1.2). *For a tensor field $F(F\neq 0)$ satisfying* (1.1) *and the operators s and t defined by* (1.2), *we have*

$$F^{K-2}s = sF^{K-2} = F^{K-2}, \quad F^{K-2}t = tF^{K-2} = 0; \tag{1.7}$$

$$F^{K-1}s = F^{K-1}, \qquad F^{K-1}t = 0. \tag{1.8}$$

PROOF. The proof follows by virtue of the equations (1.1) and (1.2).

Theorem (1.3). *For a tensor field $F(F\neq 0)$ satisfying* (1.1) *and the operators s and t defined by* (1.2), *we have*

$$Fs = sF = -(-1)^{1/2(K-1)}F^{K-2}, \quad Ft = tF = F+(-1)^{1/2(K-1)}F^{K-2}; \tag{1.9}$$

$$F^2s = -s, \quad F^2t = F^2+(-1)^{1/2(K-1)}F^{K-1}. \tag{1.10}$$

PROOF. The proof is obvious.

Corollary (1.1). *An $F(K, K-2)$-structure of maximal rank is an almost complex structure.*

PROOF. If the rank of F is maximal, $r=n$. Then $\dim S=n$ and $\dim T=0$. In this case $t=0$ and $s=I$. Thus F satisfies

$$I-(-1)^{1/2(K-1)}F^{K-1} = 0. \tag{1.11}$$

Applying F twice to (1.11) and using (1.1), we obtain

$$F^2+(-1)^{1/2(K-1)}F^{K-1} = 0,$$

which in view of (1.11) yields

$$F^2+I = 0.$$

Hence the result.

Corollary (1.2). *An $F(K, K-2)$-structure of minimal rank is an $F(K-2)$-structure.*

PROOF. If the rank of F is minimal, $2r=n$. Then $s=0$. Thus F satisfies $F^{k-1}=0$ and therefore $F^k=0$. Hence in view of (1.1), $F^{k-2}=0$. Following the general nomenclature, we call such a structure an $F(K-2)$-structure.

2. The Nijenhuis tensor of an $F(K, K-2)$-structure

Let F be an $F(K, K-2)$-structure of rank r when K is odd. Then the Nijenhuis tensor $N(X, Y)$ of F is

(2.1) $$N(X, Y) = [FX, FY]-F[FX, Y]-F[X, FY]+F^2[X, Y].$$

Therefore in consequence of (1.9) and (2.1), we have

(2.2) $$N(sX, sY) = [-(-1)^{1/2(K-1)}F^{K-2}X, -(-1)^{1/2(K-1)}F^{K-2}Y]- -F[-(-1)^{1/2(K-1)}F^{K-2}X, sY]-F[sX, -(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[sX, sY],$$

(2.3) $$N(sX, tY) = [-(-1)^{1/2(K-1)}F^{K-2}X, FY+(-1)^{1/2(K-1)}F^{K-2}Y]- -F[-(-1)^{1/2(K-1)}F^{K-2}X, tY]-F[sX, FY+(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[sX, tY],$$

(2.4) $$N(tX, sY) = [FX+(-1)^{1/2(K-1)}F^{K-2}X, -(-1)^{1/2(K-1)}F^{K-2}Y]- -F[FX+(-1)^{1/2(K-1)}F^{K-2}X, sY]-F[tX, -(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[tX, sY],$$

(2.5) $$N(tX, tY) = [FX+(-1)^{1/2(K-1)}F^{K-2}X, FY+(-1)^{1/2(K-1)}F^{K-2}Y]- -F[FX+(-1)^{1/2(K-1)}F^{K-2}X, tY]-F[tX, FY+(-1)^{1/2(K-1)}F^{K-2}Y]+F^2[tX, tY].$$

Equations (2.2), (2.3), (2.4) and (2.5), in consequence of (1.3) and (2.1), yield

(2.6) $$N(X, Y) = N(sX, sY)+N(sX, tY)+N(tX, sY)+N(tX, tY).$$

If the distribution S is integrable, $N(sX, sY)$ is exactly the Nijenhuis tensor of $F/S \overset{\text{def}}{=} F_S$. If the distribution T is integrable, $N(tX, tY)$ is exactly the Nijenhuis tensor of $F/T \overset{\text{def}}{=} F_T$.

Let $\mathcal{L}_Y F$ be the Lie derivative of the tensor field F with respect to a vector field Y. Then we have [2]

(2.7) $$(\mathcal{L}_Y F)X = F[X, Y]-[FX, Y],$$

where $\mathcal{L}_Y F$ is a tensor field of the same type as F.

Now in view of (2.1) and (2.7), we get

(2.8) $$N(sX, tY) = F(\mathcal{L}_{tY} F)\, sX-(\mathcal{L}_{FtY} F)\, sX$$

and

(2.9) $$N(tX, sY) = F(\mathcal{L}_{sY} F)\, tX-(\mathcal{L}_{FsY} F)\, tX.$$

3. Integrability conditions

In this section we shall obtain the partial integrability conditions of the $F(K, K=2)$-structure when K is odd.

Theorem (3.1). *For any two vector fields X and Y, the following hold:*

(i) *the distribution S is integrable if and only if $t\cdot N(sX, sY)=0$;*
(ii) *the distribution T is integrable if and only if $s\cdot N(tX, tY)=0$.*

PROOF. We know that for any two vector fields X and Y, the distributions S and T are integrable if and only if $t[sX, sY]=0$ and $s[tX, tY]=0$ respectively [2]. Thus in view of (1.6), (1.7), (1.9) and (2.1), the theorem follows.

Theorem (3.2). *For any two vector fields X and Y, the distributions S and T are both integrable if and only if*

$$N(X, Y) = s\cdot N(sX, sY)+N(sX, tY)+N(tX, sY)+t\cdot N(tX, tY). \tag{3.1}$$

PROOF. In consequence of (1.3), equation (2.6) can be expressed as

$$\begin{aligned} N(X, Y) = s\cdot N(sX, sY)+t\cdot N(sX, sY)+N(sX, tY)+ \\ +N(tX, sY)+s\cdot N(tX, tY)+t\cdot N(tX, tY). \end{aligned} \tag{3.2}$$

Now the result follows from equation (3.2) and theorem (3.1).

Theorem (3.3). *If the distribution S is integrable, a necessary and sufficient condition for the almost complex structure defined by $F/S=F_S$ on each integral manifold of S to be integrable is that for any two vector fields X and Y*

$$N(sX, sY) = 0, \tag{3.3}$$

which is equivalent to

$$s\cdot N(sX, sY) = 0. \tag{3.4}$$

PROOF. Suppose that the distribution S is integrable, then F induces on each integral manifold of S an almost complex structure. The induced structure is integrable if and only if its Nijenhuis tensor vanishes identically. Thus the theorem follows.

Definition (3.1). We say that an $F(K, K-2)$-structure is "s_K-partially integrable" if the distribution S is integrable and the almost complex structure F_S induced by F on each integral manifold of S is also integrable.

Theorem (3.4). *A necessary and sufficient condition for an $F(K, K-2)$-structure to be s_K-partially integrable is that for any two vector fields X and Y,*

$$N(sX, sY) = 0. \tag{3.5}$$

PROOF. Follows from theoresm (3.1) (i) and (3.3).

Theorem (3.5). *If the distribution T is integrable, a necessary and sufficient condition for the $F(K-2)$-structure defined by $F/T=F_T$ on each integral manifold of T to be integrable is that, for any two vector fields X and Y*

$$N(tX, tY) = 0, \tag{3.6}$$

which is equivalent to

$$t\cdot N(tX, tY) = 0. \tag{3.7}$$

PROOF. The proof follows from the pattern of the proof of theorem (3.3).

Definition (3.2). We say that an $F(K, K-2)$-structure is "t_K-partially integrable" if the distribution T is integrable and the $F(K-2)$-structure F_T induced by F on each integral manifold of T is also integrable.

Theorem (3.6). *A necessary and suffcient condition for the $F(K, K-2)$-structure to be t_K-partially integrable is that for any two vector fields X and Y,*

$$N(tX, tY) = 0. \tag{3.8}$$

PROOF. Follows from theorems (3.1) (ii) and (3.5).

Definition (3.3). We say that an $F(K, K-2)$-structure is "partially integrable" if and only if it is s_K-partially integrable and t_K-partially integrable, simultaneously.

Theorem (3.7). *A necessary and sufficient condition for the $F(K, K-2)$-structure to be partially integrable is that for any two vector fields X and Y,*

$$N(X, Y) = N(sX, tY)+N(tX, sY). \tag{3.9}$$

PROOF. The theorem follows by virtue of the equations (2.6), (3.5) and (3.8).

4. Conditions $N(sX, tY)=0$ and $N(tX, sY)=0$

In this section, we shall obtain the integrability conditions of an $F(K, K-2)$-structure by means of the conditions $N(sX, tY)=0$ and $N(tX, sY)=0$ when K is odd.

Theorem (4.1). *The tensor field $s(\mathscr{L}_{tY}F)s$ vanishes identically if and only if for any vector fields X and Y,*

$$N(sX, tY) = 0. \tag{4.1}$$

PROOF. In consequence of (2.8), we have $N(sX, tY)=0$ if and only if $F(\mathscr{L}_{tY}F)sX=(\mathscr{L}_{FtY}F)sX$. Thus, if $N(sX, tY)=0$, we get

$$\begin{aligned}(-1)^{1/2(K-1)}F^{K-1}(\mathscr{L}_{tY}F)sX &= (-1)^{1/2(K-1)}F^{K-2}(\mathscr{L}_{FtY}F)sX,\\ &= (-1)^{1/2(K-1)}F^{K-3}(\mathscr{L}_{F^2tY}F)sX,\\ &\dots\\ &\dots\\ &= (-1)^{1/2(K-1)}F^{K-K}(\mathscr{L}_{F^{K-1}tY}F)sX,\\ &= 0,\end{aligned}$$

in consequence of (1.8).

That is, in view of (1.2), the tensor field $s(\mathscr{L}_{tY}F)s$ vanishes identically for any vector field Y.

Theorem (4.2). *The tensor field $t(\mathscr{L}_{sY}F)t$ vanishes identically if and only if for any vector fields X and Y,*

$$N(tX, sY) = 0. \tag{4.2}$$

PROOF. The proof follows from the pattern of the proof of theorem (4.1).

When the distributions S and T are both integrable, we can choose a local coordinate system such that all S are represented by putting $(2n-2r)$ local coordinates constant and all T by putting the other $(2r-n)$ coordinates constant. Let us call such a coordinate system an "adapted coordinate system".

It can be supposed that in an adapted coordinate system, the projection operators s and t have components of the form

$$s = \begin{bmatrix} I_{2r-n} & 0 \\ 0 & 0 \end{bmatrix}, \quad t = \begin{bmatrix} 0 & 0 \\ 0 & I_{2n-2r} \end{bmatrix} \tag{4.3}$$

respectively, where I_{2r-n} is a unit matrix of order $(2r-n)$ and I_{2n-2r} is that of order $(2n-2r)$.

Since the distributions S and T are integrable, $FS \subset S$ and $FT \subset T$. Therefore, the tensor F has components of the form

$$F = \begin{bmatrix} F_{2r-n} & 0 \\ 0 & F_{2n-2r} \end{bmatrix} \tag{4.4}$$

in an adapted coordinate system, where F_{2r-n} and F_{2n-2r} are square matrices of order $(2r-n)\times(2r-n)$ and $(2n-2r)\times(2n-2r)$ respectively.

Thus, the Lie derivative $\mathscr{L}_{tY} F$ has components of the form

$$\mathscr{L}_{tY} F = \begin{bmatrix} L' & 0 \\ 0 & L'' \end{bmatrix}, \tag{4.5}$$

for any vector field tY on T.

Theorem (4.3). *Suppose that the two distributions S and T are both integrable and that an adapted coordinate system has been chosen. A necessary and sufficient condition for the local components F_{2r-n} of the $F(K, K-2)$-structure to be functions independent of the coordinates which are constant along the integral manifolds of S is that*

$$N(sX, tY) = 0 \tag{4.6}$$

for any two vector fields X and Y.

Proof. Let us suppose that $N(sX, tY)=0$ for any two vector fields X and Y. Therefore by theorem (4.1), the tensor field $s(\mathscr{L}_{tY} F)s$ vanishes identically for any vetor field Y. Hence $L'=0$. It follows that the components F_{2r-n} of the $F(K, K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of the distribution S in an adapted coordinate system.

Conversely, if the components F_{2r-n} of the $F(K, K-2)$-structure are independent of these coordinates, then $L'=0$. Therefore the tensor field $s(\mathscr{L}_{tY} F)s$ vanishes identically for any vector field Y. Hence $N(sX, tY)=0$ for any two vector fields X and Y.

Theorem (4.4). *Under the assumptions of Theorem* (4.3), *a necessary and sufficient condition for the local components F_{2r-2n} of the $F(K, K-2)$-structure to be functions independent of the coordinates which are constant along the integral manifolds of T*

is that

$$N(tX, sY) = 0, \tag{4.7}$$

for any two vector fields X and Y.

PROOF. The proof is similar to that of theorem (4.3).

Definition (4.1). We say that an $F(K, K-2)$-structure is 'integrable' if

(i) the $F(K, K-2)$-structure is partially integrable;
(ii) the components F_{2r-n} of the $F(K, K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of S in an adapted coordinate system;
(iii) the components F_{2n-2r} of the $F(K, K-2)$-structure are independent of the coordinates which are constant along the integral manifolds of T in an adapted coordinate system.

Theorem (4.5). *In order that the $F(K, K-2)$-structure be integrable, it is necessary and sufficient that*

$$N(X, Y) = 0, \tag{4.8}$$

for any two vector fields X and Y.

PROOF. The theorem follows from theorems (3.7), (4.3) and (4.4).

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