

Invariants of special semi-symmetric Finsler connection transformations

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§ 1. Introduction

Let M be an n -dimensional differentiable manifold of class C^∞ and let (x^i, y^i) be the canonical coordinates of a point $y \in T(M)$, where $T(M)$ is the tangent bundle of M [3] and let π be the canonical projection. The natural basis of $T(M)$ with respect to canonical coordinates is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ and the mapping $N: y \in T(M) \rightarrow N_y \in T(M)_y$ is a regular distribution on $T(M)$, such that: $T(M)_y = N_y \oplus T(M)_y^v$. Let $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k \frac{\partial}{\partial y^k}$ be a local basis of the n dimensional local distribution N , where $N_i^k(x, y)$ are called the coefficients of the non-linear connection defined by N . The notions and notations of M. MATSUMOTO [2] and R. MIRON [3] are used.

Let $FG = (N, F, C)$ be a Finsler connection with the coefficients (N_j^i, F_k^i, C_{jk}^i) . $\mathcal{T}$ the group of general Finsler connection transformations $t: (N, F, C) \rightarrow (\bar{N}, \bar{F}, \bar{C})$, and $\mathcal{T}_N = \{t | t \in \mathcal{T}; t = t(0, B, D)\}$ the subgroup of $\mathcal{T}$, formed by the transformations $t: (N, F, C) \rightarrow (\bar{N} = N, \bar{F}, \bar{C})$, which preserve the non-linear connection N . The transformations from $\mathcal{T}_N$ have the form:

$$(1.1) \quad \bar{N}_j^i = N_j^i; \quad \bar{F}_{jk}^i = F_{jk}^i - B_{jk}^i; \quad \bar{C}_{jk}^i = C_{jk}^i - D_{jk}^i$$

where $B, D \in Z_2^1(M)$ are arbitrary Finsler tensor fields [3]. In the following we denote by $|,|$ and $\|, \|$ the h - and v -covariant derivatives relative to (N, F, C) and to $(\bar{N}, \bar{F}, \bar{C})$ respectively.

For a Finsler connection $FG = (N, F, C)$ we can define the notion of semi-symmetric Finsler connection, analogously with the definition of semi-symmetric linear-connection. We consider the h - and v -Finsler torsion tensors:

$$(1.2) \quad T_{jk}^i = F_{jk}^i - F_{kj}^i; \quad S_{jk}^i = C_{jk}^i - C_{kj}^i$$

and the associated tensors:

$$(1.3) \quad I_{jk}^i = T_{jk}^i - \frac{1}{(n-1)} (\delta_j^i T_k - \delta_k^i T_j); \quad I_{jk}^i = S_{jk}^i - \frac{1}{(n-1)} (\delta_j^i S_k - \delta_k^i S_j)$$

where: $T_k = T_{ik}^i$ and $S_k = S_{ik}^i$ are the h - and v -torsion Finsler covectors respectively. A study of these tensors relative to its projective transformations are given in [6].

A Finsler connection $FG=(N, F, C)$ is called a semi-symmetric Finsler connection, if $I_{jk}^i=0$ and $I_{jk}^i=0$.

This condition is equivalent with the existence of an 1-form on $T(M)$:

$$(1.4) \quad \tau = \tau_j dx^j + \omega_k \delta y^k$$

such that:

$$(1.5) \quad T_k = (1-n)\tau_k; \quad S_k = (1-n)\omega_k$$

For a fixed semi-symmetric Finsler connection $FG=(N, F, C)$ the h -torsion tensor and the v -torsion tensor have the form:

$$(1.6) \quad T_{jk}^i = \tau_j \delta_k^i - \tau_k \delta_j^i; \quad S_{jk}^i = \omega_j \delta_k^i - \omega_k \delta_j^i$$

In general, for a Finsler connection $FG=(N, F, C)$ we have the following three curvature tensors:

$$(1.7) \quad \begin{aligned} S_{jkh}^i &= \frac{\partial C_{jk}^i}{\partial y^h} - \frac{\partial C_{jh}^i}{\partial y^k} + C_{jk}^r C_{rh}^i - C_{jh}^r C_{rk}^i \\ R_{jkh}^i &= \frac{\delta F_{jk}^i}{\delta x^h} - \frac{\delta F_{jh}^i}{\delta x^k} + F_{jk}^r F_{rh}^i - F_{jh}^r F_{rk}^i + C_{jr}^i R_{kh}^r \\ P_{jkh}^i &= \frac{\partial F_{jk}^i}{\partial y^h} - C_{jh|k}^i + C_{jr}^i P_{kh}^r \end{aligned}$$

where:

$$(1.8) \quad R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j}; \quad P_{jk}^i = \frac{\partial N_j^i}{\partial y^k} - F_{kj}^i$$

We define also the Finsler tensor:

$$(1.9) \quad K_{jkh}^i \stackrel{\text{def}}{=} R_{jkh}^i - C_{jr}^i R_{kh}^r$$

where R_{kh}^r is the curvature tensor of the non-linear connection N_j^i , given by the first relation of (1.8).

If the non-linear connection N is integrable, then: $K_{jkh}^i = R_{jkh}^i$. We denote by:

$$(1.10) \quad s_{kh} = S_{ikh}^i; \quad r_{kh} = R_{ikh}^i; \quad B_{kh} = K_{ikh}^i$$

the tensors of Bianchi type and by:

$$(1.11) \quad S_{jk} = S_{jki}^i; \quad R_{jk} = R_{jki}^i; \quad K_{jk} = K_{jki}^i$$

the tensors of Ricci type, and use also the notations:

$$(1.12) \quad (h \operatorname{div} T)_{ij} \stackrel{\text{def}}{=} T_{ij|k}^k; \quad (v \operatorname{div} S)_{ij} \stackrel{\text{def}}{=} S_{ij|k}^k$$

If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, then between the Finsler tensors of Ricci type and of Bianchi type we have the relations [1]:

$$(1.13) \quad K_{jk} - K_{kj} = B_{kj} + (n-2)(h \operatorname{div} T)_{kj}$$

$$(1.14) \quad S_{jk} - S_{kj} = s_{kj} + (n-2)(v \operatorname{div} S)_{kj}$$

In the present paper we study invariants of those Finsler connection transformations, which transform semi-symmetric Finsler connections in semi-symmetric Finsler connections again. First we study invariants obtained by the use of S and R curvature tensors, and afterwards invariants obtained using the third curvature tensor P . Finally the properties of the Finsler 1-form $\omega \in A(T(M))$ associate to the Finsler connection transformation are given.

§ 2. Invariants of S and R type

In this section the invariants obtained by the use of the curvature tensors S and R (or K) are established.

The most general Finsler connection transformation $t \in \mathcal{T}_N$, which has the invariants I_1 and I_2 , has been studied in [8] obtaining the following result:

Lemma 2.1. *The set $\mathcal{T}_{NI}$ of all Finsler connections, which preserves the linear connection N and has the invariants I_1 and I_2 , is given by:*

$$(2.1) \quad \bar{N}_j^i = N_j^i; \quad \bar{F}_{jk}^i = F_{jk}^i + \delta_j^i \beta_k - \delta_k^i \beta_j - U_{jk}^i; \quad \bar{C}_{jk}^i = C_{jk}^i + \delta_j^i \alpha_k - \delta_k^i \alpha_j - V_{jk}^i$$

where $FG=(N, F, C)$ is a fixed Finsler connection; α_k and β_k are arbitrary Finsler covectors, and $U, V \in Z_2^1(M)$ are arbitrary Finsler tensor fields of type $(1, 2)$, with the properties $U_{jk}^i = U_{kj}^i$; $V_{jk}^i = V_{kj}^i$.

The most general case of the Finsler connection transformations $t \in \mathcal{T} = \{t | t: (N, F, C) \rightarrow (\bar{N}, \bar{F}, \bar{C})\}$ with the invariant tensors I_1 and I_2 has been studied in [7].

If we have $I_1=0$ and $I_2=0$, then from (2.1) it follows $\bar{I}_1=0$, $\bar{I}_2=0$, and reciprocally. Thus $\mathcal{T}_{NI}$ contains also the set of the general semi-symmetric connection transformations.

Afterwards we consider the special case of the Finsler connection transformations (2.1) with $U=0$ and $V=0$. In this case we obtain:

$$(2.2) \quad \bar{N}_j^i = N_j^i; \quad \bar{F}_{jk}^i = F_{jk}^i + \delta_j^i \beta_k - \delta_k^i \beta_j; \quad \bar{C}_{jk}^i = C_{jk}^i + \delta_j^i \alpha_k - \delta_k^i \alpha_j$$

where $FG=(N, F, C)$ is a semi-symmetric Finsler connection. Then also $F\bar{F}=(\bar{N}, \bar{F}, \bar{C})$ is a semi-symmetric Finsler connection.

Between the curvature tensor K_{jkh}^i , S_{jkh}^i and $\bar{K}_{jkh}^i$, $\bar{S}_{jkh}^i$ respectively we obtain the relations:

$$(2.3) \quad \bar{K}_{jkh}^i = K_{jkh}^i + \delta_j^i (\varphi_{kh} - \varphi_{hk}) - \delta_k^i \varphi_{jh} + \delta_h^i \varphi_{jk}$$

$$(2.4) \quad \bar{S}_{jkh}^i = S_{jkh}^i + \delta_j^i (\Phi_{kh} - \Phi_{hk}) - \delta_k^i \Phi_{jh} + \delta_h^i \Phi_{jk}$$

where we have:

$$(2.5) \quad \Phi_{kh} = \alpha_k|_h + \alpha_k \alpha_h - \alpha_k \omega_h; \quad \varphi_{kh} = \beta_k|_h + \beta_k \beta_h - \beta_k \tau_h$$

From (2.3) and (2.4) follows the relations:

$$(2.6) \quad \bar{K}_{jk} = K_{jk} + (n-2) \varphi_{jk} + \varphi_{kj}; \quad B_{kh} = \bar{B}_{kh} + (n-1)(\varphi_{kh} - \varphi_{hk})$$

$$(2.7) \quad \bar{S}_{jk} = S_{jk} + (n-2) \Phi_{jk} + \Phi_{kj}; \quad s_{kh} = \bar{s}_{kh} + (n-1)(\Phi_{kh} - \Phi_{hk})$$

From (2.6) and (2.7) we have:

$$(2.8) \quad \bar{K}_{jk} - \bar{K}_{kj} = K_{jk} - K_{kj} + (n-3)(\varphi_{jk} - \varphi_{kj}); \quad \bar{B}_{kh} + \bar{B}_{hk} = B_{kh} + B_{hk}$$

$$(2.9) \quad \bar{S}_{jk} - \bar{S}_{kj} = S_{jk} - S_{kj} + (n-3)(\Phi_{jk} - \Phi_{kj}); \quad \bar{s}_{kh} - \bar{s}_{hk} = s_{kh} - s_{hk}$$

Thus it follows the:

Proposition 2.1. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, then the transformation (2.2) has the invariants:*

$$(2.10) \quad \bar{K}_{jk} - \bar{K}_{kj} - \frac{n-3}{n-1} \bar{B}_{jk} = K_{jk} - K_{kj} - \frac{n-3}{n-1} B_{jk}; \quad \bar{B}_{kh} + \bar{B}_{hk} = B_{kh} + B_{hk}$$

$$(2.11) \quad \bar{S}_{jk} - \bar{S}_{kj} - \frac{n-3}{n-1} \bar{s}_{jk} = S_{jk} - S_{kj} - \frac{n-3}{n-1} s_{jk}; \quad \bar{s}_{kh} + \bar{s}_{hk} = s_{kh} + s_{hk}$$

From (2.6) and (2.7) we have:

$$(2.12) \quad \varphi_{jk} = \frac{1}{n-1} (\bar{K}_{jk} - K_{jk}) + \frac{1}{(n-1)^2} (\bar{B}_{jk} - B_{jk})$$

$$(2.13) \quad \Phi_{jk} = \frac{1}{n-1} (\bar{S}_{jk} - S_{jk}) + \frac{1}{(n-1)^2} (\bar{s}_{jk} - s_{jk})$$

From (2.1), (2.2), (2.4), (2.5), (2.8) and (2.9) follow the relations:

$$(2.14) \quad \bar{A}_{1jkh}^i \stackrel{\text{def}}{=} \bar{K}_{jkh}^i + \frac{1}{n-1} \delta_j^i \bar{B}_{hk} + \delta_k^i \left(\frac{\bar{K}_{jh}}{n-1} + \frac{\bar{B}_{jh}}{(n-1)^2} \right) - \delta_h^i \left(\frac{\bar{K}_{jk}}{n-1} + \frac{\bar{B}_{jk}}{(n-1)^2} \right) = \\ = K_{jkh}^i + \frac{1}{n-1} \delta_j^i B_{hk} + \delta_k^i \left(\frac{K_{jh}}{n-1} + \frac{B_{jh}}{(n-1)^2} \right) - \delta_h^i \left(\frac{K_{jk}}{n-1} + \frac{B_{jk}}{(n-1)^2} \right) \stackrel{\text{def}}{=} A_{1jkh}^i$$

$$(2.15) \quad \bar{A}_{2jkh}^i \stackrel{\text{def}}{=} \bar{S}_{jkh}^i + \frac{1}{n-1} \delta_j^i \bar{s}_{hk} + \delta_k^i \left(\frac{\bar{S}_{jh}}{n-1} + \frac{\bar{s}_{jh}}{(n-1)^2} \right) - \delta_h^i \left(\frac{\bar{S}_{jk}}{n-1} + \frac{\bar{s}_{jk}}{(n-1)^2} \right) = \\ = S_{jkh}^i + \frac{1}{n-1} \delta_j^i s_{hk} + \delta_k^i \left(\frac{S_{jh}}{n-1} + \frac{s_{jh}}{(n-1)^2} \right) - \delta_h^i \left(\frac{S_{jk}}{n-1} + \frac{s_{jk}}{(n-1)^2} \right) \stackrel{\text{def}}{=} A_{2jkh}^i$$

Thus we have the following:

Theorem 2.1. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, then the transformation (2.2) has the invariants A_1 and A_2 given by (2.14) and (2.15) respectively.*

From (2.5) it follows:

$$(2.16) \quad \Phi_{kh} - \Phi_{hk} = \frac{\partial \alpha_k}{\partial y^h} - \frac{\partial \alpha_h}{\partial y^k}; \quad \varphi_{kh} - \varphi_{hk} = \frac{\partial \beta_k}{\partial x^h} - \frac{\partial \beta_h}{\partial x^k}$$

Thus we have the:

Lemma 2.1. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, then:*

$$(2.17) \quad \Phi_{kh} - \Phi_{hk} = 0 \quad \text{iff} \quad \frac{\partial \alpha_k}{\partial y^h} = \frac{\partial \alpha_h}{\partial y^k}; \quad \varphi_{kh} - \varphi_{hk} = 0 \quad \text{iff} \quad \frac{\delta \beta_k}{\delta x^h} - \frac{\delta \beta_h}{\delta x^k} = 0,$$

From this Lemma and the relations (2.6), (2.7), (2.8), (2.9) it follows the:

Proposition 2.2. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, then*

(a) *the Finsler—Bianchi tensors B_{kh} and s_{kh} are invariants of the transformation (2.2) if and only if:*

$$(2.18) \quad \frac{\partial \alpha_h}{\partial y^k} = \frac{\partial \alpha_k}{\partial y^h}; \quad \frac{\partial \beta_h}{\delta x^k} = \frac{\partial \beta_k}{\delta x^h}$$

(b) *the Finsler tensors $K_{jk} - K_{kj}$ and $S_{jk} - S_{kj}$ are invariants of the transformation (2.2) if and only if the relations (2.18) hold.*

From (2.12) and (2.18) it follows:

$$(2.19) \quad \varphi_{jk} = \frac{1}{n-1} (\bar{K}_{jk} - K_{jk}); \quad \Phi_{jk} = \frac{1}{n-1} (\bar{S}_{jk} - S_{jk})$$

From (2.3), (2.4) and (2.19) we have the relations:

$$(2.20) \quad \begin{aligned} \bar{B}_{1jkh} &\stackrel{\text{def}}{=} \bar{K}_{jkh}^i - \frac{1}{n-1} (\delta_h^i \bar{K}_{jk} - \delta_k^i \bar{K}_{jh}) = \\ &= K_{jkh}^i - \frac{1}{n-1} (\delta_h^i K_{jk} - \delta_k^i K_{jh}) \stackrel{\text{def}}{=} B_{1jkh}^i \end{aligned}$$

$$(2.21) \quad \begin{aligned} \bar{B}_{2jkh}^i &\stackrel{\text{def}}{=} \bar{S}_{jkh}^i - \frac{1}{n-1} (\delta_h^i \bar{S}_{jk} - \delta_k^i \bar{S}_{jh}) = \\ &= S_{jkh}^i - \frac{1}{n-1} (\delta_h^i S_{jk} - \delta_k^i S_{jh}) \stackrel{\text{def}}{=} B_{2jkh}^i \end{aligned}$$

Reciprocally: From (2.20) and (2.21), using (2.3)—(2.4) and (2.6)—(2.7) we have:

$$(2.22) \quad \varphi_{kh} - \varphi_{hk} = 0; \quad \Phi_{kh} - \Phi_{hk} = 0$$

Theorem 2.2. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, then the necessary and sufficient condition that the transformation (2.2) has the invariants of Weyl-type (2.20)—(2.21) is given by (2.18).*

We associate to the connection $FG=(N, F, C)$ the Finsler tensors of projective Weyl-type:

$$(2.23) \quad W_{1jkh}^i = K_{jkh}^i + \frac{1}{n+1} \delta_j^i B_{hk} + \delta_k^i \left(\frac{K_{jh}}{n-1} + \frac{B_{jh}}{n^2-1} \right) - \delta_h^i \left(\frac{K_{jk}}{n-1} + \frac{B_{jk}}{n^2-1} \right)$$

$$(2.24) \quad W_{2jkh}^i = S_{jkh}^i + \frac{1}{n+1} \delta_j^i s_{hk} + \delta_k^i \left(\frac{S_{jh}}{n-1} + \frac{s_{jh}}{n^2-1} \right) - \delta_h^i \left(\frac{S_{jk}}{n-1} + \frac{s_{jk}}{n^2-1} \right)$$

analogously with the projective tensors of Weyl from the linear connection theory.

The projective transformations of the semi-symmetric connections, characterized by the invariants $\bar{W}_1$ and $\bar{W}_2$ are studied in [4] and [5]. In the case of the transformations (2.2) we have:

(2.25)

$$\bar{W}_1^i = W_1^i + \frac{2}{n+1} \delta_j^i (\varphi_{hk} - \varphi_{kh}) + \frac{2}{n^2-1} \delta_k^i (\varphi_{jh} - \varphi_{hj}) - \frac{2}{n^2-1} \delta_h^i (\varphi_{jk} - \varphi_{kj})$$

(2.26)

$$\bar{W}_2^i = W_2^i + \frac{2}{n+1} \delta_j^i (\Phi_{hk} - \Phi_{kh}) + \frac{2}{n^2-1} \delta_k^i (\Phi_{jh} - \Phi_{hj}) - \frac{2}{n^2-1} \delta_h^i (\Phi_{jk} - \Phi_{kj})$$

From Lemma 2.1 and the relations (2.23)—(2.24) it follows the:

Theorem 2.3. *If $F\Gamma = (N, F, C)$ is a semi-symmetric Finsler connection, then the necessary and sufficient condition that the transformation (2.2) has the invariants $\bar{W}_1$ and $\bar{W}_2$ is given by the relations (2.18).*

Corollary 2.1. *If $\bar{B}_1 = B_1$, $\bar{B}_2 = B_2$ then and only then we have:*

$$\bar{W}_1 = W_1 \quad \text{and} \quad \bar{W}_2 = W_2.$$

From (1.13)—(1.14) and (2.23)—(2.24) we have:

$$(2.27) \quad W_1^i = \frac{n-2}{n-1} (h \operatorname{div} T)_{kh}; \quad W_2^i = \frac{n-2}{n-1} (v \operatorname{div} S)_{kh}$$

Thus it follows the:

Proposition 2.3. *The necessary and sufficient condition that for a semi-symmetric Finsler connection $F\Gamma = (N, F, C)$ the Finsler tensors W_1^i, W_2^i vanish, is that the torsional divergence of $F\Gamma = (N, F, C)$ vanishes.*

From (2.25)—(2.26) it follows:

$$(2.28) \quad \bar{W}_1^i = W_1^i + 2(n-2)(\varphi_{kh} - \varphi_{hk}); \quad \bar{W}_2^i = W_2^i + 2(n-2)(\Phi_{kh} - \Phi_{hk})$$

From the Proposition 2.3 and the relation (2.22), (2.28) it follows the:

Proposition 2.4. *The necessary and sufficient condition that a semi-symmetric Finsler connection $F\Gamma = (N, F, C)$ with null torsional divergence be transformed by the transformation (2.2) in a semi-symmetric Finsler connection $F\bar{\Gamma} = (N, \bar{F}, \bar{C})$ with the same property, is that the transformation has the property (2.18).*

Next it follows the:

Proposition 2.5. If $F\Gamma=(N, F, C)$ and $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ are semi-symmetric Finsler connections with a null torsional divergence, then we have:

$$(2.29) \quad K_{jk}-K_{kj}=B_{kj}; \quad S_{jk}-S_{kj}=s_{kj}; \quad \bar{K}_{jk}-\bar{K}_{kj}=\bar{B}_{kj}; \quad \bar{S}_{jk}-\bar{S}_{kj}=\bar{s}_{kj}$$

and it follows that B, B, A, A, W, W are invariants for the transformation (2.2).

Definition 2.1. The semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ is equiaffine, if $B_{kj}=0, s_{kj}=0$.

Proposition 2.6. If $F\Gamma=(N, F, C)$ is an equiaffine, semi-symmetric Finsler connection, with null torsional divergence, then the Finsler connection $F\bar{\Gamma}=(\bar{N}, \bar{F}, \bar{C})$ obtained by the transformation (2.2) has the same properties. In this case the invariants A, B, W coincide and the invariants A, B, W coincide also.

Proposition 2.7. If $F\Gamma=(N, F, C)$ is a semi-symmetric Finsler connection, $\bar{K}_{jkh}^i=0, \bar{S}_{jkh}^i=0$ and α_n, β_n satisfies the condition (2.18), then $F\Gamma$ is equiaffine Finsler connection with null torsional divergence.

In this case we have:

$$(2.30) \quad S_{jkh}^i = \frac{1}{n-1} (\delta_h^i S_{jk} - \delta_k^i S_{jh}); \quad K_{jkh}^i = \frac{1}{n-1} (\delta_h^i K_{jk} - \delta_k^i K_{jh})$$

$$(2.31) \quad S_{jk} = S_{kj}; \quad K_{jk} = K_{kj}$$

From (2.30) it follows:

$$(2.32) \quad S_{jkh|l}^i = \frac{1}{n-1} (\delta_h^i S_{jk|l} - \delta_k^i S_{jh|l}); \quad S_{jkh|i}^i = \frac{1}{n-1} (S_{jk|h} - S_{jh|k})$$

$$(2.33) \quad K_{jkh|l}^i = \frac{1}{n-1} (\delta_h^i K_{jk|l} - \delta_k^i K_{jh|l}); \quad K_{jkh|i}^i = \frac{1}{n-1} (K_{jk|h} - K_{jh|k})$$

and we obtain the Bianchi identities:

$$(2.34) \quad S_{jkh}^i + S_{kjh}^i + S_{hjk}^i = 0; \quad K_{jkh}^i + K_{khj}^i + K_{hjk}^i = 0$$

Thus it follows the:

Proposition 2.8. For a semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ the Bianchi identities (2.34) holds if and only if $F\Gamma$ is with null torsional divergence.

The second group of Bianchi identities for a semi-symmetric Finsler connection $F\Gamma=(N, F, C)$ is given in [5]:

$$(2.35) \quad \sum_{(jkh)} K_{ijk|h}^r = 2 \sum_{(jkh)} K_{ijk}^r \tau_h - \frac{\partial F_{ij}^r}{\partial y^s} R_{kh}^s; \quad \sum_{(jkh)} S_{ijk|h}^r = 2 \sum_{(jkh)} S_{ijk}^r \omega_h$$

From (2.32) and (2.35) it follows:

$$(2.36) \quad S_{ij|k} - S_{ik|j} = 2S_{ir} S_{kj}^r$$

If N is integrable we obtain analogously:

$$(2.37) \quad K_{ij|k} - K_{ik|j} = 2K_{ir}T_{kj}^r$$

Proposition 2.9. *If $F\Gamma = (N, F, C)$ is an equiaffin semi-symmetric Finsler connection with null torsional divergence, then the relations (2.30)—(2.36) hold, and if moreover N is integrable, then the relations (2.30)—(2.35) and (2.37) hold however.*

Consequently if the conditions of Propositions 2.7 and 2.9 are satisfied, then we have the following Propositions:

Proposition 2.10. *If $F\Gamma = (N, F, C)$ is a symmetric Finsler connection and $F\bar{\Gamma} = (\bar{N}, \bar{F}, \bar{C})$ is a projectively flate Finsler connection ($\bar{W}_1 = 0, \bar{W}_2 = 0$), then $F\Gamma$ is an equiaffine and projectively flate Finsler connection.*

Definition 2.1. A Finsler connection $F\Gamma = (N, F, C)$ is called a Finsler F -connection, if:

$$(2.38) \quad \frac{\partial F_{jk}^i}{\partial y^r} R_{hi}^r + \frac{\partial F_{jh}^i}{\partial y^r} R_{ik}^r + \frac{\partial F_{ji}^i}{\partial y^s} R_{kh}^s = 0$$

Proposition 2.11. *If $F\Gamma = (N, F, C)$ is a symmetric Finsler connection and we have $\bar{K}_{jkh}^i = 0, \bar{S}_{jkh}^i = 0$, then $F\Gamma$ is equiaffine, projectively flate and the tensor $S_{ijk} \stackrel{\text{def}}{=} S_{ij|k} - S_{ik|j}$ is symmetric. The tensor $K_{ijk} \stackrel{\text{def}}{=} K_{j|k} - K_{k|j}$ is symmetric if and only if $F\Gamma$ is a Finsler F -connection.*

If $\bar{W}_{1jkh|r}^i = 0, \bar{W}_{2jkh|r}^i = 0$, it follows $\bar{W}_{1ikh|r}^i = 0, \bar{W}_{2ikh|r}^i = 0$ and from (2.28) we have:

$$(2.39) \quad (\varphi_{kh} - \varphi_{hk})|_r = 0; \quad (\Phi_{kh} - \Phi_{hk})|_r = 0$$

From (2.25), (2.26) and (2.39) we have:

$$(2.40) \quad W_{1jkh|r}^i = 0; \quad W_{2jkh|r}^i = 0$$

Thus it follows the:

Proposition 2.12. *If $F\Gamma = (N, F, C)$ is a semi-symmetric Finsler connection and $\bar{W}_{1jkh|r}^i = 0, \bar{W}_{2jkh|r}^i = 0$, then also $W_{1jkh|r}^i = 0, W_{2jkh|r}^i = 0$.*

Particularly we have the:

Proposition 2.13. *If $F\Gamma = (N, F, C)$ is a symmetric Finsler connection and $F\bar{\Gamma} = (\bar{N}, \bar{F}, \bar{C})$ is a projective symmetric Finsler connection ($\bar{W}_{1jkh|r}^i = 0, \bar{W}_{2jkh|r}^i = 0$), then also $F\Gamma$ is projective symmetric ($W_{1jkh|r}^i = 0, W_{2jkh|r}^i = 0$).*

§ 3. Invariants of $\mathcal{P}$ type

In this section the invariants of the transformations (2.2) obtained from the third curvature tensor P , or from the tensor $\mathcal{P}$ are studied.

The third curvature tensor P_{jkh}^i and the tensor:

$$(3.1) \quad \mathcal{P}_{jkh}^i \stackrel{\text{def}}{=} \left(P_{jkh}^i - C_{jr}^i \frac{\partial N_k^r}{\partial y^h} \right) - \left(P_{jkh}^i - C_{jr}^i \frac{\partial N_h^r}{\partial y^k} \right)$$

are given in [3].

By a Finsler connection transformation (2.2) we obtain:

$$(3.2) \quad \bar{\mathcal{P}}_{jkh}^i = \mathcal{P}_{jkh}^i + \delta_j^i (\Theta_{kh} - \Theta_{hk}) - \delta_k^i \Theta_{jh} + \delta_h^i \Theta_{jk}$$

where:

$$(3.3) \quad \Theta_{kh} = \beta_{k|h} + \alpha_k |h + \beta_k \alpha_h + \alpha_k \beta_h - \beta_k \omega_h - \alpha_k \tau_h$$

From (3.2) it follows:

$$(3.4) \quad \bar{\mathcal{P}}_{jk} = \mathcal{P}_{jk} + (n-1) \Theta_{jk} - (\Theta_{jk} - \Theta_{kj}); \quad \bar{\pi}_{kh} = \pi_{kh} + (n-1)(\Theta_{kh} - \Theta_{hk})$$

where:

$$\mathcal{P}_{jk} = \mathcal{P}_{jki}^i; \quad \pi_{kh} = \mathcal{P}_{ikh}^i; \quad \bar{\mathcal{P}}_{jk} = \bar{\mathcal{P}}_{jki}^i; \quad \bar{\pi}_{kh} = \bar{\mathcal{P}}_{ikh}^i$$

and it follows also:

$$(3.5) \quad \bar{\mathcal{P}}_{jk} - \bar{\mathcal{P}}_{kj} = \mathcal{P}_{jk} - \mathcal{P}_{kj} + (n-3)(\Theta_{jk} - \Theta_{kj}); \quad \bar{\pi}_{kh} - \pi_{kh} = -(\bar{\pi}_{hk} - \pi_{hk})$$

From (3.2)–(3.5) we obtain the invariant $\bar{A}_{3jkh}^i = A_{3jkh}^i$, given by:

$$(3.6) \quad \begin{aligned} \bar{A}_{3jkh}^i &\stackrel{\text{def}}{=} \bar{\mathcal{P}}_{jkh}^i + \frac{1}{n-1} \delta_j^i \bar{\pi}_{hk} + \delta_k^i \left(\frac{\bar{\mathcal{P}}_{jh}}{n-1} + \frac{\bar{\pi}_{jh}}{(n-1)^2} \right) - \delta_h^i \left(\frac{\bar{\mathcal{P}}_{jk}}{n-1} + \frac{\bar{\pi}_{jk}}{(n-1)^2} \right) = \\ &= \mathcal{P}_{jkh}^i + \frac{1}{n-1} \delta_j^i \pi_{hk} + \delta_k^i \left(\frac{\mathcal{P}_{jh}}{n-1} + \frac{\pi_{jh}}{(n-1)^2} \right) - \delta_h^i \left(\frac{\mathcal{P}_{jk}}{n-1} + \frac{\pi_{jk}}{(n-1)^2} \right) \stackrel{\text{def}}{=} A_{3jkh}^i \end{aligned}$$

It follows the:

Theorem 3.1. *The Finsler connection transformation (2.2) between the semi-symmetric Finsler connections $FG=(N, F, C)$ and $F\bar{G}=(\bar{N}, \bar{F}, \bar{C})$ has the invariant A_{3jkh}^i .*

For $n \geq 3$ the relations are equivalents:

$$(a) \Theta_{kh} = \Theta_{hk}; \quad (b) \bar{\pi}_{jk} = \pi_{jk}; \quad (c) \bar{\mathcal{P}}_{jk} - \bar{\mathcal{P}}_{kj} = \mathcal{P}_{jk} - \mathcal{P}_{kj}.$$

It follows the:

Theorem 3.2. *If $F\Gamma = (N, F, C)$ is a semi-symmetric Finsler connection, then a necessary and sufficient condition that the transformation (2.2) has the invariant:*

$$(3.7) \quad \bar{B}_{jkh}^i \stackrel{\text{def}}{=} \bar{\mathcal{P}}_{jkh}^i - \frac{1}{n-1} (\delta_h^i \bar{\mathcal{P}}_{jk} - \delta_k^i \bar{\mathcal{P}}_{jh}) = \mathcal{P}_{jkh}^i - \frac{1}{n-1} (\delta_h^i \mathcal{P}_{jk} - \delta_k^i \mathcal{P}_{jh}) \stackrel{\text{def}}{=} B_{jkh}^i$$

is given by: $\Theta_{kh} = \Theta_{kh}$.

We associate to $\mathcal{P}_{jkh}^i$ a Finsler tensor W_{jkh}^i analogous with the Weyl projective curvature tensor where:

$$(3.8) \quad W_{jkh}^i \stackrel{\text{def}}{=} \mathcal{P}_{jkh}^i + \frac{1}{n+1} \delta_j^i \pi_{hk} + \delta_k^i \left(\frac{\mathcal{P}_{jh}}{n-1} + \frac{\pi_{jh}}{n^2-1} \right) - \delta_h^i \left(\frac{\mathcal{P}_{jk}}{n-1} + \frac{\pi_{jk}}{n^2-1} \right)$$

We obtain:

$$(3.9) \quad \bar{W}_{jkh}^i = W_{jkh}^i + \frac{2}{n+1} \delta_j^i (\Theta_{kh} - \Theta_{hk}) - \frac{2}{n^2-1} [\delta_k^i (\Theta_{jh} - \Theta_{hj}) - \delta_h^i (\Theta_{jk} - \Theta_{kj})]$$

and:

$$(3.10) \quad \bar{W}_{ikh}^i = W_{ikh}^i + 2 \frac{n-2}{n-1} (\Theta_{kh} - \Theta_{hk}); \quad \bar{W}_{jki}^i = W_{jki}^i - 0.$$

It follows the:

Theorem 3.3. *The necessary and sufficient condition that the Finsler connection transformation (2.2) between the semi-symmetric Finsler connections $F\Gamma = (N, F, C)$ and $\bar{F}\Gamma = (\bar{N}, \bar{F}, \bar{C})$ has the invariant $\bar{W}_{jkh}^i = W_{jkh}^i$, is given by: $\Theta_{kh} = \Theta_{hk}$.*

In this case the invariant $\bar{B}_{jkh}^i = B_{jkh}^i$ is an outcome of the invariant W_{jkh}^i .

§ 4. Properties of the Finsler 1-form $\omega \in \Lambda(T(M))$, associate to the Finsler connection transformation

Let $\omega \in \Lambda(T(M))$ be the Finsler 1-form

$$(4.1) \quad \omega = \beta_k dx^k + \alpha_i \delta y^i$$

associate to the Finsler connection transformation (2.2). We denote:

$$(4.2) \quad \Theta_k = \beta_k + N_k^i \alpha_i$$

$$(4.3) \quad \alpha_{kh} = \frac{\delta \beta_k}{\delta x^h} - \frac{\delta \beta_h}{\delta x^k} + R_{kh}^i \alpha_i$$

$$(4.4) \quad \beta_{kh} = \frac{\partial \Theta_k}{\partial y^h} - \frac{\partial \Theta_h}{\partial y^k}$$

$$(4.5) \quad \gamma_{kh} = \frac{\partial \alpha_k}{\partial y^h} - \frac{\partial \alpha_h}{\partial y^k}$$

Thus it follows the:

Theorem 4.1. *The necessary and sufficient condition that the Finsler 1-form ω be closed ($d\omega=0$) is given by: $\alpha_{kh}=0$, $\beta_{kh}=0$, $\gamma_{kh}=0$.*

PROOF. The exterior differential of ω is given by:

$$(4.6) \quad d\omega = \frac{1}{2} \alpha_{kh} dx^h \wedge dx^k + \alpha_{kh} \delta y^h \wedge dx^k + \frac{1}{2} \alpha_{kh} \delta y^h \wedge \delta y^k$$

where:

$$(4.7) \quad \alpha_{kh} = \beta_{k|h} - \beta_{h|k} + T_{kh}^s \beta_s + R_{kh}^i \alpha_i$$

$$(4.8) \quad \alpha_{kh} = \beta_{k|h} - \alpha_{h|k} + C_{kh}^s \beta_s + P_{kh}^s \alpha_s$$

$$(4.9) \quad \alpha_{kh} = \alpha_{k|h} - \alpha_{h|k} + S_{kh}^s \alpha_s$$

It follows:

$$(4.10) \quad \alpha_{kh} = \alpha_{kh}, \quad \alpha_{kh} = \beta_{kh} - N_k^s \gamma_{sh}; \quad \alpha_{kh} = \gamma_{kh}$$

Then $d\omega=0$ if and only if $\alpha_{kh}=0$, $\beta_{kh}=0$, $\gamma_{kh}=0$.

We have also the:

Theorem 4.2. *Let $FG=(N, F, C)$ be a semi-symmetric Finsler connection. The necessary and sufficient condition that B_{kh} and $K_{kh} - K_{hk}$ be invariants of the transformation (2.2), is given by:*

$$(4.11) \quad d\omega = \frac{1}{2} R_{kh}^s dx^h \wedge dx^k + \alpha_{kh} \delta y^h \wedge dx^k + \frac{1}{2} \alpha_{kh} \delta y^h \wedge \delta y^k$$

or by:

$$(4.12) \quad d\omega = \alpha_{kh} \delta y^h \wedge dx^k + \frac{1}{2} \alpha_{kh} \delta y^h \wedge \delta y^k$$

in case if N is integrable.

From Theorem 2.2 and 1.2 it follows:

Theorem 4.3. *The necessary and sufficient condition that a Finsler connection transformation (2.2), where $FG=(N, F, C)$ is a semi-symmetric Finsler connection, has the invariant B_{jk}^i is given by (4.11) or by (4.12) if N is integrable.*

From (4.5), (2.1) and (2.5) it follows the:

Theorem 4.4. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection then the necessary and sufficient condition that B_{jk}^i be an invariant of the transformation (2.2) is given by:*

$$(4.13) \quad d\omega = \frac{1}{2} \alpha_{kh} dx^h \wedge dx^k + \alpha_{kh} \delta y^h \wedge dx^k$$

or by

$$(4.14) \quad d\omega = \frac{1}{2} \alpha_{kh} dx^h \wedge dx^k + \beta_{kl} \delta y^l \wedge dx^k$$

if N is integrable.

From the two above Theorems it follows the:

Theorem 4.5. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, the necessary and sufficient condition that $\bar{B}_1$ and $\bar{B}_2$ are invariants by a transformation (2.2) is given by:*

$$(4.15) \quad d\omega = \frac{1}{2} R_{kh}^s dx^h \wedge dx^k + \beta_{kh} \delta y^h \wedge dx^k$$

or by:

$$(4.16) \quad d\omega = \beta_{kh} \delta y^h \wedge dx^k$$

if N is integrable.

From (3.3) and (4.4) it follows the relation:

$$(4.17) \quad \beta_{kh} - \beta_{hk} - (N_k^s \gamma_{sh} - N_h^s \gamma_{sk}) - \left(\frac{\partial N_k^i}{\partial y^h} - \frac{\partial N_h^i}{\partial y^k} \right) \alpha_i = \Theta_{kh} - \Theta_{hk}$$

Thus we have the:

Theorem 4.6. *If $FG=(N, F, C)$ is a semi-symmetric Finsler connection, the necessary and sufficient condition that $\bar{B}_1, \bar{B}_2$ and $\bar{B}_3$ are invariants by a transformation (2.2), is that $d\omega$ is of the form (4.14) and β_{kh} satisfies the relation:*

$$(4.18) \quad \beta_{kh} - \beta_{hk} = \left(\frac{\partial N_k^i}{\partial y^h} - \frac{\partial N_h^i}{\partial y^k} \right) \alpha_i$$

or that $d\omega$ be of the form (4.15), and β_{kh} satisfies the relation (4.17), if N integrable.

If ω is closed ($d\omega=0$), then the Finsler covectors α_k and β_k have the form $\alpha_k = \varrho|_k$, $\beta_k = \varrho|_k$, where ϱ is a Finsler function, while ω is locally exact ($\omega=d\varrho$). In this case from the above theorems it follows:

Theorem 4.7. *If ω is a closed Finsler 1-form, then the necessary and sufficient condition that $\bar{B}_1 = \bar{B}_1, \bar{B}_2 = \bar{B}_2, \bar{B}_3 = \bar{B}_3$ is that $\alpha_k = \varrho|_k$ satisfies the conditions:*

$$R_{kh}^s \alpha_s = 0; \quad \left(\frac{\partial N_k^i}{\partial y^h} - \frac{\partial N_h^i}{\partial y^k} \right) \alpha_i = 0.$$

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