

Some applications of the generalized hypergeometric function involving certain subclasses of analytic functions *

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Abstract

Two familiar subclasses of analytic functions are considered here: the class $\mathcal{S}(\alpha)$ of analytic functions $f(z)$ satisfying the inequality

$$\operatorname{Re} \{f(z)/z\} > \alpha \quad (0 \leq \alpha < 1),$$

and the class $\mathcal{R}$ of analytic functions whose derivative has a positive real part. The object of this paper is to present several interesting applications of the generalized hypergeometric function, which involve the classes $\mathcal{S}(\alpha)$ and $\mathcal{R}$, and the concept of subordination between analytic functions. A theorem on the radius of univalence for a certain class of generalized hypergeometric functions is also established.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be in the class $\mathcal{S}(\alpha)$ if it satisfies the following inequality:

$$(1.2) \quad \operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > \alpha \quad (z \in \mathcal{U})$$

for $0 \leq \alpha < 1$.

The class $\mathcal{S}(\alpha)$ was introduced by GOEL [5], and was studied subsequently by CHEN ([2], [3]). In particular, the class $\mathcal{S}(0)$ was studied by GOEL [6] and YAMAGUCHI [18].

A function $f(z)$ belonging to the class $\mathcal{A}$ is said to be in the class $\mathcal{R}$ if it satisfies the following inequality:

$$(1.3) \quad \operatorname{Re} \{f'(z)\} > 0 \quad (z \in \mathcal{U}).$$

The class $\mathcal{R}$ was introduced by MACGREGOR [11].

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In order to recall the concept of subordination between analytic functions, let $f(z)$ and $g(z)$ be analytic in the unit disk $\mathcal{U}$. The function $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $h(z)$ analytic in the unit disk $\mathcal{U}$, with $h(0)=0$ and $|h(z)| < 1$, such that

$$(1.4) \quad f(z) = g(h(z))$$

for $z \in \mathcal{U}$. We denote this subordination by

$$(1.5) \quad f(z) \prec g(z).$$

In particular, if $g(z)$ is univalent in the unit disk $\mathcal{U}$, the subordination (1.5) is equivalent to (cf. [4], p. 190).

$$(1.6) \quad f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

The concept of subordination can be traced back to LINDELÖF [8], although LITTLEWOOD ([9], [10]) and ROGOSINSKI ([14], [15]) introduced the term and established the basic results involving subordination. More recently, SUFFRIDGE [17], and HALLENBECK and RUSCHEWEYH [7], studied various families of subordinate functions.

Finally, let α_j ($j=1, \dots, p$) and β_j ($j=1, \dots, q$) be complex numbers with

$$\beta_j \neq 0, -1, -2, \dots; \quad j = 1, \dots, q.$$

Also let $(\lambda)_n$ denote the Pochhammer symbol defined by

$$(1.7) \quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\dots(\lambda+n-1), & \text{if } n \in \mathcal{N} = \{1, 2, 3, \dots\}. \end{cases}$$

Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by (cf., e.g., [16], p. 33 *et seq.*)

$$(1.8) \quad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(\beta_1)_n \dots (\beta_q)_n} \frac{z^n}{n!}$$

$$(p \leq q+1).$$

It may be recalled that the ${}_pF_q(z)$ series in (1.8) converges absolutely for $|z| < \infty$ if $p < q+1$, and for $z \in \mathcal{U}$ if $p = q+1$. Furthermore, if we set

$$(1.9) \quad \omega = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j,$$

then the ${}_pF_q(z)$ series (1.8), with $p = q+1$, is absolutely convergent for

$$(1.10) \quad |z| = 1 \quad \text{if} \quad \operatorname{Re}(\omega) > 0,$$

and conditionally convergent for

$$(1.11) \quad |z| = 1, \quad z \neq 1, \quad \text{if} \quad -1 < \operatorname{Re}(\omega) \leq 0.$$

We now introduce the following class of generalized hypergeometric functions:

Definition. The generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) is said to be in the class $\mathcal{A}(p; q; \alpha)$ if it satisfies the following inequality:

$$(1.12) \quad \operatorname{Re} \{ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \} > \alpha \quad (z \in \mathcal{U})$$

for $0 \leq \alpha < 1$.

Making use of the above definitions, we present several interesting applications of the generalized hypergeometric function ${}_pF_q(z)$, which involve the classes $\mathcal{S}(\alpha)$ and $\mathcal{R}$, and the concept of subordination between analytic functions. We also establish a theorem on the radius of univalence for the above class of generalized hypergeometric functions.

2. Application involving subordination between analytic functions

Our first application of the generalized hypergeometric function ${}_pF_q(z)$ depends upon a result due to Nehari [12, p. 168], which we recall here as

Lemma 1. Let $\varphi(z)$ be analytic in the unit disk $\mathcal{U}$ and satisfy $|\varphi(z)| \leq 1$ for $z \in \mathcal{U}$. Then

$$(2.1) \quad |\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad (z \in \mathcal{U}).$$

By using Lemma 1 and the concept of subordination, we shall prove

Theorem 1. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p; q; \alpha)$. Then

$$(2.2) \quad |{}_pF_q(\alpha_1 + 1, \dots, \alpha_p + 1; \beta_1 + 1, \dots, \beta_q + 1; z)| \leq \left(\left| \prod_{j=1}^q \beta_j \right| / \left| \prod_{j=1}^p \alpha_j \right| \right) \left(\frac{2(1-\alpha)}{(1-|z|)^2} \right),$$

where

$$(2.3) \quad \prod_{j=1}^p \alpha_j \prod_{j=1}^q \beta_j \neq 0.$$

The result (2.2) is sharp.

PROOF. We note that

$$(2.4) \quad \operatorname{Re} \left(\frac{1 + (1-2\alpha)z}{1-z} \right) > \alpha \quad (z \in \mathcal{U}).$$

Hence, by virtue of the definition of subordination, we obtain

$$(2.5) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \prec \frac{1 + (1-2\alpha)z}{1-z}$$

for $z \in \mathcal{U}$. Thus we may write

$$(2.6) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \frac{1 + (1-2\alpha)w(z)}{1-w(z)},$$

where $w(z)$ is analytic in the unit disk $\mathcal{U}$, with

$$(2.7) \quad w(0) = 0 \quad \text{and} \quad |w(z)| < 1.$$

Differentiating both sides of (2.6), we get

$$(2.8) \quad {}_pF_q'(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \frac{2(1-\alpha) w'(z)}{[1-w(z)]^2}.$$

Note that

$$(2.9) \quad {}_pF_q'(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \\ = \left(\prod_{j=1}^p \alpha_j / \prod_{j=1}^q \beta_j \right) {}_pF_q(\alpha_1+1, \dots, \alpha_p+1; \beta_1+1, \dots, \beta_q+1; z),$$

and that

$$(2.10) \quad |w(z)| \leq |z| \quad \text{for } z \in \mathcal{U},$$

by the Schwarz lemma.

Applying Lemma 1 to $w(z)$, we find that

$$(2.11) \quad |{}_pF_q(\alpha_1+1, \dots, \alpha_p+1; \beta_1+1, \dots, \beta_q+1; z)| \leq \\ \leq \left(\prod_{j=1}^q \beta_j / \prod_{j=1}^p \alpha_j \right) \cdot \frac{2(1-\alpha)}{[1-|w(z)|]^2} \cdot \frac{1-|w(z)|^2}{1-|z|^2} \leq \left(\prod_{j=1}^q \beta_j / \prod_{j=1}^p \alpha_j \right) \left(\frac{2(1-\alpha)}{(1-|z|)^2} \right)$$

provided that the condition (2.3) holds true.

Finally, by taking the generalized hypergeometric function defined by

$$(2.12) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \frac{1+(1-2\alpha)z}{1-z},$$

we readily verify that the result (2.2) is sharp.

3. Application involving the class $\mathcal{S}(\alpha)$

We need the following result given by CHEN [3]:

Lemma 2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}(\alpha)$. Then, for $0 \leq |z| < \frac{1}{2}$,*

$$(3.1) \quad \operatorname{Re} \{f'(z)\} \geq \frac{1+2(2\alpha-1)|z|+(2\alpha-1)|z|^2}{(1+|z|)^2},$$

and, for $\frac{1}{2} \leq |z| < 1$,

$$(3.2) \quad \operatorname{Re} \{f'(z)\} \geq \frac{\alpha-2\alpha|z|^2+(2\alpha-1)|z|^4}{(1-|z|^2)^2}.$$

The results (3.1) and (3.2) are sharp.

By using Lemma 2, we now prove

Theorem 2. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p; q; \alpha)$. Then, for $0 \leq |z| < \frac{1}{2}$,

$$(3.3) \quad \operatorname{Re} \{ {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 2; \beta_1, \dots, \beta_q, 1; z) \} \cong \frac{1 + 2(2\alpha - 1)|z| + (2\alpha - 1)|z|^2}{(1 + |z|)^2},$$

and, for $\frac{1}{2} \leq |z| < 1$,

$$(3.4) \quad \operatorname{Re} \{ {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 2; \beta_1, \dots, \beta_q, 1; z) \} \cong \frac{\alpha - 2\alpha|z|^2 + (2\alpha - 1)|z|^4}{(1 - |z|^2)^2}.$$

The results (3.3) and (3.4) are sharp.

PROOF. Define a function $H(z)$ by

$$(3.5) \quad H(z) = z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$$

for $z \in \mathcal{U}$. Since ${}_pF_q(z)$ is in the class $\mathcal{A}(p; q; \alpha)$, we have

$$(3.6) \quad \operatorname{Re} \left\{ \frac{H(z)}{z} \right\} > \alpha \quad (z \in \mathcal{U}),$$

which implies that $H(z) \in \mathcal{S}(\alpha)$.

Applying Lemma 2 to $H(z)$, we find that

$$(3.7) \quad \operatorname{Re} \{ H'(z) \} \cong \frac{1 + 2(2\alpha - 1)|z| + (2\alpha - 1)|z|^2}{(1 + |z|)^2}$$

for $0 \leq |z| < \frac{1}{2}$, and that

$$(3.8) \quad \operatorname{Re} \{ H'(z) \} \cong \frac{\alpha - 2\alpha|z|^2 + (2\alpha - 1)|z|^2}{(1 - |z|^2)^2}$$

for $\frac{1}{2} \leq |z| < 1$.

Now it is not difficult to verify that

$$(3.9) \quad \begin{aligned} H'(z) &= z {}_pF_q'(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) + {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = \\ &= {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 2; \beta_1, \dots, \beta_q, 1; z), \end{aligned}$$

which, in conjunction with (3.7) and (3.8), yields the assertions (3.3) and (3.4) of Theorem 2.

Finally, by taking the generalized hypergeometric function defined by

$$(3.10) \quad \begin{aligned} &{}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) * {}_1F_0(2; -; z) = \\ &= \begin{cases} \frac{1 + 2(\alpha - 1)z + (2\alpha - 1)z^2}{(1 + z)^2} & \left(0 \leq |z| < \frac{1}{2} \right), \\ \frac{\alpha - 2\alpha z^2 + (2\alpha - 1)z^4}{(1 - z^2)^2} & \left(\frac{1}{2} \leq |z| < 1 \right), \end{cases} \end{aligned}$$

we can show that the results (3.3) and (3.4) are sharp; here $f(z) * g(z)$ denotes the Hadamard product (or convolution) of the functions $f(z)$ and $g(z)$.

4. Application involving the class $\mathcal{R}$

In order to apply the generalized hypergeometric function ${}_pF_q(z)$ to the class $\mathcal{R}$, we require the following lemma due to MACGREGOR [11]:

Lemma 3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{R}$. Then*

$$(4.1) \quad |a_n| \leq \frac{2}{n} \quad (n \geq 2),$$

$$(4.2) \quad |f'(z)| \leq \frac{1+|z|}{1-|z|} \quad (z \in \mathcal{U}),$$

$$(4.3) \quad \operatorname{Re} \{f'(z)\} \geq \frac{1-|z|}{1+|z|} \quad (z \in \mathcal{U}),$$

$$(4.4) \quad |f(z)| \leq -|z| + 2 \log(1+|z|) \quad (z \in \mathcal{U}),$$

and

$$(4.5) \quad |f(z)| \leq -|z| - 2 \log(1-|z|) \quad (z \in \mathcal{U}).$$

We now establish

Theorem 3. *Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p; q; \alpha)$. Then*

$$(4.6) \quad \left| \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \right| \leq 2(1-\alpha) \cdot n! \quad (n \geq 1),$$

$$(4.7) \quad |{}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)| \leq \frac{1+(1-2\alpha)|z|}{1-|z|} \quad (z \in \mathcal{U}),$$

$$(4.8) \quad \operatorname{Re} \{ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \} \geq \frac{1-(1-2\alpha)|z|}{1+|z|} \quad (z \in \mathcal{U}),$$

$$(4.9) \quad |{}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 1; \beta_1, \dots, \beta_q, 2; z) - \alpha| \leq (\alpha-1) + \frac{2(1-\alpha) \log(1+|z|)}{|z|} \\ (z \in \mathcal{U} - \{0\}),$$

and

$$(4.10) \quad |{}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 1; \beta_1, \dots, \beta_q, 2; z) - \alpha| \leq (\alpha-1) + \frac{2(\alpha-1) \log(1-|z|)}{|z|} \\ (z \in \mathcal{U} - \{0\}).$$

The results (4.6) to (4.10) are sharp.

PROOF. We introduce a function $G(z)$ defined by

$$(4.11)$$

$$G(z) = \int_0^z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; t) dt = {}_{p+1}F_{q+1}(\alpha_1, \dots, \alpha_p, 1; \beta_1, \dots, \beta_q, 2; z)$$

for $z \in \mathcal{U}$. It follows from (4.11) that

$$(4.12) \quad \operatorname{Re} \{G'(z)\} = \operatorname{Re} \{ {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \} > \alpha.$$

Next we define a function $G_1(z)$ by

$$(4.13) \quad G_1(z) = \frac{G(z) - \alpha z}{1 - \alpha}$$

for $z \in \mathcal{U}$. Then it is easily verified that

$$(4.14) \quad \operatorname{Re} \{G'_1(z)\} = \operatorname{Re} \left\{ \frac{G'(z) - \alpha}{1 - \alpha} \right\} > 0, \quad z \in \mathcal{U};$$

which, by the definition (1.3), implies that

$$G_1(z) \in \mathcal{R}.$$

Noting that

$$(4.15) \quad G_1(z) = z + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n (1)_n}{(\beta_1)_n \dots (\beta_q)_n (2)_n (1 - \alpha)} z^{n+1},$$

and applying Lemma 3 to $G_1(z)$, we immediately get the assertions (4.6) to (4.10) of Theorem 3.

Finally, taking the generalized hypergeometric function defined by

$$(4.16) \quad {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) = 1 + 2(1 - \alpha) \sum_{n=1}^{\infty} z^n,$$

we see that each of the results (4.6) to (4.10) is sharp.

5. Univalence of the generalized hypergeometric function

CARLSON and SHAFFER [1] presented a study of various interesting classes of starlike, convex, and prestarlike hypergeometric functions by applying a linear operator defined by a certain convolution. Recently, Owa and Srivastava [13] derived several interesting results concerning univalent generalized hypergeometric functions, starlike generalized hypergeometric functions of order α , and convex generalized hypergeometric functions of order α . In this section we determine the radius of univalence for the generalized hypergeometric functions belonging to the class $\mathcal{A}(p; q; \alpha)$ with the aid of the following lemma due to CHEN [3]:

Lemma 4. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{S}(\alpha)$ with $0 \leq \alpha < \frac{1}{10}$. Then $f(z)$ is univalent in $|z| < r_1$, where r_1 is given by*

$$(5.1) \quad r_1 = \sqrt{\frac{2(1-\alpha)}{1-2\alpha}} - 1.$$

The result is sharp.

By using Lemma 4, we shall derive

Theorem 4. Let the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.8) belong to the class $\mathcal{A}(p; q; \alpha)$ with $0 \leq \alpha < \frac{1}{10}$. Then the function

$$z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z)$$

is univalent in $|z| < r_1$, where r_1 is given by (5.1). The result is sharp.

PROOF. The hypothesis that ${}_pF_q(z)$ is in the class $\mathcal{A}(p; q; \alpha)$ implies that

$$(5.2) \quad z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \in \mathcal{S}(\alpha),$$

and the proof of Theorem 4 follows easily from Lemma 4.

The assertion of Theorem 4 is sharp for the generalized hypergeometric function defined by (3.10).

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