

On additive arithmetical functions with values in the circle group

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1. We shall use the following standard notations: $\mathbf{N}$ =natural numbers; $\mathbf{Z}$ =rational integers; $\mathbf{Q}_x$ =multiplicative group of positive rationals; $\mathbf{R}_x$ =multiplicative group of positive reals; $\mathbf{Q}$ =additive group of rationals; $\mathbf{R}$ =additive group of reals; T =one-dimensional circle group (torus), each of them in the usual topology.

Let G be an Abelian group, metrically compact in addition. A function $\varphi: \mathbf{N} \rightarrow G$ is called an additive numbertheoretical function if $\varphi(nm) = \varphi(n) + \varphi(m)$ holds for every coprime pairs of m and n , $m, n \in \mathbf{N}$. We shall say that φ is completely additive if $\varphi(mn) = \varphi(m) + \varphi(n)$ for any $m, n \in \mathbf{N}$.

Let x_v ($v=1, 2, \dots$) be an infinite sequence in G . We shall say that it is of property D , if for any convergent subsequence x_{v_l} ($l=1, 2, \dots$) the shifted subsequence x_{v_l+1} ($l=1, 2, \dots$) is convergent too. We shall say that x_v is of property K if $x_{v+1} - x_v \rightarrow 0$ ($v \rightarrow \infty$). It is obvious that the sequences having property K all have property D and that the second set is much wider in general than the first one. In our paper [1] we proved that if $x_v = \varphi(v)$, φ is completely additive, then the fulfilment of D implies the fulfilment of K . This, by a recent result due to E. Wirsing allowed us to give a complete characterization of the completely additive functions having property D .

Our purpose in this paper is to solve this problem for the additive functions if $G=T$.

2. Theorem 1. *Let $\varphi: \mathbf{N} \rightarrow T$ be an additive function satisfying property D , i.e. the existence of $\lim \varphi(n_k) = g$ ($n_1 < n_2 < \dots$) implies the existence of $\lim \varphi(n_k + 1) = g'$. Then*

$$(2.1) \quad \varphi(n) = u(n) + \tau \log n \pmod{1}$$

with a suitable $\tau \in \mathbf{R}$, u is an additive function, $u(m) = 0$ for every odd m , $u(2^\alpha) = u(2)$ for every $\alpha \geq 1$.

Furthermore, if $\tau \neq 0$, then $u(2^\alpha) = 0$ or $1/2$.

PROOF. Let $\varphi \in D$ be fixed in the sequel. Let S denote the set of accumulation points of the sequence $a_n := \varphi(n)$ ($n \in \mathbf{N}$), i.e. let $g \in S$ if there exists a subsequence $n_1 < n_2 < \dots$ of natural numbers such that $g = \lim_k \varphi(n_k) = g$, then let

$$F(g) := g' = \lim_k \varphi(n_k + 1).$$

Lemma 1. *The correspondence $F: S \rightarrow S$ is a function and $F(S) = S$.*

The proof of it was given in [1].

Let $p(n)$ and $P(n)$ denote the smallest and the largest prime factor of $n \in \mathbb{N}$. We shall say that a sequence $M := \{M_1 < M_2 < \dots\}$ ($M_v \in \mathbb{N}$) belongs to $\mathcal{M}_0$ if for every $d \in \mathbb{N}$ $M_v - 1 \equiv 0 \pmod{d}$ whenever v is sufficiently large, $v > v_0(d, M)$. We shall say that a sequence $L := \{L_1 < L_2 < \dots\}$ belongs to $\mathcal{M}_k$ ($k \in \mathbb{Z}$) if $M (= L_{-k}) = \{L_1 - k < L_2 - k < \dots\}$ belongs to $\mathcal{M}_0$. Let $L := \{L_1 < L_2 < \dots\} \in \mathcal{M}_k$. We shall say that $L \in \tilde{\mathcal{M}}_k$ ($\subseteq \mathcal{M}_k$) if there exists $\lim_v \varphi(L_v) := a(L)$.

Let H_k denote the set of the limit points $a(L)$ while L runs over $\tilde{\mathcal{M}}_k$.

First we consider H_0 . Let $M \in \tilde{\mathcal{M}}_0$, $R = \{R_1 < R_2 < \dots\} \in \tilde{\mathcal{M}}_0$. For a given v let j_v be so large that $p(R_{j_v}) > P(M_v)$, $j_v > j_{v-1}$; $j_0 = 0$. Then $(R_{j_v}, M_v) = 1$, consequently $\varphi(R_{j_v} M_v) = \varphi(R_{j_v}) + \varphi(M_v)$, and the sequence $\{S_v = R_{j_v} M_v \ (v = 1, 2, \dots)\}$ belongs to $\tilde{\mathcal{M}}_0$. Hence it follows immediately that $\lim \varphi(M_v) = g_1$, $\lim \varphi(R_v) = g_2$ implies $\lim \varphi(S) = g_1 + g_2 \in H_0$. Then H_0 is a semigroup in T . It is clear that H_0 is closed. The closedness of H_0 implies that H_0 is a compact semigroup in T , and so by [4] (9.16) it must be a group. Consequently H_0 is a discrete group or $H_0 = T$.

3. Continuation of the proof, case $H_0 = T$

From Lemma 1 we get that $H_0 = H_1 = H_2 = \dots = T$.

Lemma 2. *Let $R := \{\varphi(n) | n \in \mathbb{N}\}$. Then $R + T \subseteq T$. R is everywhere dense in T .*

Let $g \in T (= H_0)$, and $M \in \tilde{\mathcal{M}}_0$ be such a sequence for which $a(M) = g$. Let $N \in \mathbb{N}$ be fixed. Then $(M_v, N) = 1$ for every large v , so $\varphi(M_v N) = \varphi(M_v) + \varphi(N) \rightarrow g + \varphi(N)$, i.e. $R + T \subseteq T$. Since $0 \in T$, therefore $R \subseteq T$. It is obvious that R is everywhere dense in T .

Lemma 3. *F is continuous on T .*

For the proof see Lemma 5 in [1].

Let $a, b \in \mathbb{N}$.

$$(3.1) \quad l(a, b) := \varphi(ab) - \varphi(a) - \varphi(b).$$

If the common prime factors of a and b are $p_1, p_2, \dots, p_r$, $a = p_1^{\alpha_1} \dots p_r^{\alpha_r} a_1$, $b = p_1^{\beta_1} \dots p_r^{\beta_r} b_1$, $(a_1, b_1) = 1$, $p_j + a_1$, $p_j + b_1$ ($j = 1, \dots, r$), then

$$(3.2) \quad l(a, b) = \sum_{j=1}^r l(p_j^{\alpha_j}, p_j^{\beta_j})$$

Let now a, b, c, d, f be nonnegative integers such that $a + b = c + d$, $ab = cd + f$. Then

$$(3.3) \quad (x+a)(x+b) = (x+c)(x+d) + f$$

is an identity.

Let $g \in T$, $n_1 < n_2 < \dots$ be a sequence in $\mathcal{M}_k$ ($k \geq 0$) such that $\varphi(n_v) \rightarrow g$.

Since $n_v - (k+1) \equiv 0 \pmod{d}$ if $v > v_0(d)$ for every d , and $(n_v + a, n_v + b) | b - a$, therefore $l(n_v + a, n_v + b) = l(a + k + 1, b + k + 1)$ for every large v , consequently

$$\varphi((n_v + a)(n_v + b)) \rightarrow l(a + k + 1, b + k + 1) + \varphi(n_v + a) + \varphi(n_v + b)$$

for every large v . Arguing similarly with c, d instead of a, b , we get that

(3.4)

$$l(a + k + 1, b + k + 1) + F^a(g) + F^b(g) = F^f[l(c + k + 1, d + k + 1) + F^c(g) + F^d(g)].$$

Hence we get that $l(c + k_1 + 1, d + k_2 + 1) = l(c + k_2 + 1, d + k_2 + 1)$ implies $l(a + k_1 + 1, b + k_1 + 1) = l(a + k_2 + 1, b + k_2 + 1)$.

Let now $q > 2$ be a prime, $(a, q) = 1$, $b - a = q$, $d = 0$, $c = a + b = 2a + q$. Then $(2a + q + 1 + k, 1 + k) = 1$ for at least one arithmetical progression $k \pmod{2a + q}$. Let us fix such an arithmetical progression $k \equiv k_0 \pmod{2a + q}$. Then $l(c + 1 + k, d + 1 + k) = 0$ as $k \equiv k_0$. Let us consider now $l(a + 1 + k, b + 1 + k)$ while $k \equiv k_0 \pmod{2a + q}$. Since $b - a = q$, $(a + 1 + k, b + 1 + k) | b - a = q$, therefore $(a + 1 + k, b + 1 + k) = 1$ or q . Since $(2a + q, q) = 1$, therefore the system $a + 1 + k \not\equiv 0 \pmod{q}$, $k \equiv k_0 \pmod{2a + q}$ is solvable. Hence we get that $l(a + 1 + k, b + 1 + k) = 0$ as $k \equiv k_0 \pmod{2a + q}$. Similarly, the congruences $a + 1 + k \equiv 0 \pmod{q^\alpha}$, $a + 1 + k \not\equiv 0 \pmod{q^{\alpha+1}}$, $b + 1 + k \not\equiv 0 \pmod{q^2}$, $k \equiv k_0 \pmod{c}$ are solvable for every $\alpha \geq 1$, we get that

$$l(q^\alpha, q) = 0 \quad (\alpha = 1, 2, \dots).$$

This implies immediately that $\varphi(q^\alpha) = \alpha \varphi(q)$. So we have proved the following

Lemma 4. $\varphi(mn) = \varphi(m) + \varphi(n)$ for every pairs of odd integers m, n .

Lemma 5. We have

$$(3.5) \quad F[g] + F^2[g] = F^2[g + F^3[g]].$$

Let n_v be so chosen that $\varphi(n_v) \rightarrow g$. Starting from the identity $(x+1)(x+2) = x(x+3) + 2$, taking into account Lemma 4, we get that

$$\varphi((n_v + 1)(n_v + 2)) \rightarrow F(g) + F^2(g),$$

$$\varphi(n_v(n_v + 3)) \rightarrow g + F^3[g], \quad \varphi(n_v(n_v + 3) + 2) \rightarrow F^2[g + F^3[g]].$$

Let now η be such an element in T for which $F^3(\eta) = 0$. The existence of such an η is almost obvious (see [1]). Then, if we put $g = \eta$ into (3.5), we get $F(\eta) = 0$, which implies

Lemma 6.

$$(3.6) \quad F^2[0] = 0.$$

Let now k be an odd integer, $M = \{M_1 < M_2 < \dots\} \in \tilde{\mathcal{M}}_0$ such that $\varphi(M_v) \rightarrow -\varphi(k)$. Then

$$\varphi(kM_v) = \varphi(k) + \varphi(M_v) \rightarrow 0,$$

$$\varphi(kM_v + k) \rightarrow F^k[0] = F[0],$$

$$\varphi(kM_v + k) = \varphi(k) + \varphi(M_v + 1) \rightarrow \varphi(k) + F[-\varphi(k)].$$

This implies that

$$(3.7) \quad F[-\varphi(k)] = -\varphi(k) + F[0].$$

In [1] we proved that F is continuous. Since $H_0 = T$, therefore $\varphi(k)$ ($k=1, 3, 5, \dots$) is everywhere dense in T . Since T is a group therefore $\{-\varphi(k)\}$ is everywhere dense as well. Consequently we have

Lemma 7.

$$(3.8) \quad F(g) = g + F[0].$$

Hence it follows immediately that

$$F^2[g] = F[F(g) + F[0]] = F^2[g] + 2F[0],$$

i.e. that

$$(3.9) \quad 2F[0] = 0.$$

So $F[0] = \gamma$, $\gamma = 0$ or $1/2$.

Lemma 8. *We have*

$$\lim_{n \rightarrow \infty} \{\varphi(n+1) - \varphi(n)\} = \gamma$$

Assuming the contrary, there exists a subsequence $n_1 < n_2 < \dots$ of positive integers such that $\Delta\varphi(n_j) = \varphi(n_j+1) - \varphi(n_j) \rightarrow h$, $h \neq \gamma$. We can choose a suitable convergent subsequence of $\varphi(n_{j_l})$, $\varphi(n_{j_l}) \rightarrow g$. Then $\varphi(n_{j_l}+1) \rightarrow g+h$, $F[g] = g+h$, $h = F[0]$. This contradicts to $F[0] = \gamma$.

Lemma 9. *If $\psi: \mathbf{N} \rightarrow T$ is additive, and $\Delta\psi(n) := \psi(n+1) - \psi(n) \rightarrow 0$ ($n \rightarrow \infty$), then ψ is completely additive.*

For the proof see e.g. [5].

We shall need the following theorem due to E. WIRSING [3] which we quote now as

Lemma 10. *If $\psi: \mathbf{N} \rightarrow T$ is completely additive and $\Delta\psi(n) \rightarrow 0$, then $\psi(n) \equiv \tau \log n \pmod{1}$ with a suitable $\tau \in R$.*

Now we finish the proof of the theorem. If $\gamma = 0$, then $F[0] = 0$, Lemma 9 and 10 implies that (2.1) holds with $u(n) = 0$ ($n = 1, 2, \dots$). Let us assume that $\gamma = 1/2$. Then $\psi(n) := 2\varphi(n)$ satisfies the conditions of Lemma 10, consequently $\psi(n) = K \log n \pmod{1}$. Consequently $\varphi(n) = \tau \log n + u(n) \pmod{1}$, $\tau = \frac{K}{2}$, where $2u(n) = 0$ for every n . Since $\Delta\varphi(n) = o(1) + \Delta u(n)$, $\Delta\varphi(n) \rightarrow 1/2$, therefore $\Delta u(n) \rightarrow 1/2$, which by $u(n) = 0$ or $1/2$ gives that $u(n+2) - u(n) = 0$ for every large n . Then $u(n)$ is ultimately periodic mod 2, completely additive on the set of odd numbers, consequently $u(n^2) = u(n)$ ($(n, 2) = 1$), i.e. $u(n) = 0$ if $(n, 2) = 1$. Furthermore $u(n+1) - u(n) = 1/2$ for every large n , consequently $u(2) = u(2^2) = \dots = 1/2$.

The proof is finished in the case $H_0 = T$.

4. Continuation of the proof, case $H_0 \neq T$

In this case H_0 is a discrete subgroup, the case $H_0 = \{0\}$ may be occur.

Let A be the set of those sequences $x = \{x_1, x_2, \dots\}$, $x_v \in \mathbb{N}$ for which there exists $a(x) = \lim_v \varphi(x_v)$, and $p(x_v) \rightarrow \infty$ as $v \rightarrow \infty$. Let A be the set of limit points $a(x)$, $x \in A$. Since $\tilde{\mathcal{M}}_0 \subseteq A$, $\tilde{\mathcal{M}}_{-2} \subseteq A$, therefore $H_{-2} \subseteq A$, $H_0 \subseteq A$. It is obvious that A is a closed semigroup, and so it is a group. To prove this we need only to repeat the argument has been done in section 2. Let now $x \in A$, $R = \{R_1 < R_2 < \dots\} \in \tilde{\mathcal{M}}_{-1}$. Let j_v be so large that $j_v > j_{v-1}$, $P(R) < p(x_{j_v})$. Then the sequence $S_v = R_v x_{j_v}$ belongs to $\tilde{\mathcal{M}}_{-1}$, $(R_v, x_{j_v}) = 1$, and so $a(R) + a(x) \in H_{-1}$. So we have

$$(4.1) \quad A + H_{-1} \subseteq H_{-1},$$

and from $H_{-2} \subseteq A$, $H_0 \subseteq A$, we get that

$$(4.2) \quad H_{-2} + H_{-1} \subseteq H_{-1}, \quad H_0 + H_{-1} \subseteq H_{-1}.$$

Let us consider now H_{-2} . Let $x, y \in \tilde{\mathcal{M}}_{-2}$, $a(x) = h_1$, $a(y) = h_2$. Then the sequence $z_v = x_v y_{j_v}$, $p(y_{j_v}) > P(x_v)$ belongs to $\tilde{\mathcal{M}}_0$, and $a(z) = a(x) + a(y)$, i.e.

$$(4.3) \quad h_1 + h_2 \in H_0, \text{ whenever } h_1, h_2 \in H_{-2}.$$

Similarly, if we take $x \in \tilde{\mathcal{M}}_{-2}$, $y \in \tilde{\mathcal{M}}_0$, we get that $z \in \mathcal{M}_{-2}$, consequently

$$(4.4) \quad h + g \in H_{-2} \text{ whenever } h \in H_{-2}, g \in H_0.$$

Since F^2 is a function, $F^2(H_{-2}) = H_0$, therefore $\text{card}(H_{-2}) \cong \text{card}(H_0)$. Let us fix h_2 , and let h to run over H_{-2} in (4.3). Then we get immediately that

$$(4.5) \quad \text{card}(H_{-2}) = \text{card}(H_0).$$

But this implies that

$$(4.6) \quad \text{card}(H_{-2}) = \text{card}(H_{-1})$$

From (4.1) we get that $\text{card}(A) \cong \text{card}(H_{-1})$. From (4.5), (4.6), and from $H_0 \cup H_2 \subseteq A$ we get that $H_{-2} = H_0$.

Let now $M = \{M_1 < M_2 < \dots\} \in \tilde{\mathcal{M}}_0$, $a(M) = g$, $k > 1$. Then $\varphi(M_v + k) \rightarrow F^k[g]$.

We may assume that $M_v - 1 \equiv 0 \pmod{(k+1)}$ for $v \geq 1$. Then the sequence $N_v = \frac{M_v - 1}{k+1} + 1$ belongs to $\mathcal{M}_0$, consequently a rarified subsequence of it belongs to $\tilde{\mathcal{M}}_0$, $\varphi(N_{j_v}) \rightarrow g^* \in H_0$. Since $\varphi(M_v + k) = \varphi((k+1)N_v)$, we get that $F^k[g] = g^* + \varphi(k+1)$, and so $H_k = F^k[H_0] \subseteq H_0 + \varphi(k+1)$. Observing that the function $F^2: H_{-2} \rightarrow H_0$ is one to one, we have that $\text{card}(H_k) = \text{card}(H_0)$ for every k , consequently

$$(4.7) \quad H_k = H_0 + \varphi(k+1).$$

Since $H_0 = H_2 = H_4 = \dots$, from (4.7) we get that $\varphi(2n+1) \in H_0$ whenever $n \in \mathbb{N}$. Furthermore, $H_1 = H_3 = \dots$ that by (4.7) implies that $\varphi(2k) - \varphi(2l) \in H_0$ for each $k, l \in \mathbb{N}$.

Since $H_{-2} = H_0$, there exists a sequence $\{y_v\} \in \tilde{\mathcal{M}}_{-2}$ such that $\varphi(y_v) \rightarrow 0$. Let $z_v = y_v - 3$, and z_{n_v} be so rarified that $\lim \varphi(z_{n_v}) = h$ there exists. Since $3 + y_v$, therefore $(z_v, y_v) = 1$ for each large v . Starting from the identity $2 + x(x+3) = (x+1)(x+2)$, substituting $x = z_{n_v}$, we deduce that

$$F^2[h + F^3[h]] = F[h] + F^2[h],$$

that by $0 = F^3[h] = \lim \varphi(y_v)$ gives that $F(h) = 0$, whence $F^2[0] = 0$ immediately follows. Since in this case $R = \{\varphi(n) | n \in \mathbb{N}\}$ is a discrete set, for a sequence $n_1 < n_2 < \dots$ $\varphi(n_v)$ is convergent if and only if $\varphi(n_v) = \varphi(n_{v+1})$ for every large v . Let $\mathcal{B}$ denote the set of all $n \in \mathbb{N}$ for which $\varphi(n) = 0$. $\mathcal{B}$ contains infinitely many element. From $F^2[0] = 0$ it follows immediately that $n \in \mathcal{B}$, n is large enough implies $n+2 \in \mathcal{B}$. Since $\mathcal{B}$ contains infinitely many odd elements, therefore $\varphi(n) = 0$ for every large odd integer. From the additivity we get that $\varphi(n) = 0$ for every odd n . Let now $n_v = 2^x v - 1$ ($v = 1, 3, \dots$).

Then $\varphi(n_v) = 0$ ($v = 1, 3, \dots$), consequently $\varphi(n_v + 1) = \varphi(2^x) = F[0]$ ($v = 1, 2, \dots$), i.e. $\varphi(2) = \varphi(2^2) = \varphi(2^3) = \dots$.

The theorem is proved.

References

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