

A variant of Kátai's minimax theorem for additive functions

By MARIJKE VAN ROSSUM-WIJSMULLER (Philadelphia, Pa)

Let $f(n)$ be a nonnegative, strongly additive function, which tends to 0 monotonically on the sequence of primes. Assume

$$(1) \quad \sum_{y < p \leq 2y} f(p) = o(1), \quad y \rightarrow \infty.$$

For $C > 0$, let $n_1 < \dots < n_t < \dots$ be the sequence of integers determined by the condition $f(n_i) \leq C$.

In this note we investigate the gaps $(n_{i+1} - n_i)$.

If $f(n)$ has a limiting distribution $F(x)$ and t is determined by

$$(2) \quad n_1 < \dots < n_t \leq x, \quad f(n_i) \leq C$$

then
$$\lim_{x \rightarrow \infty} \frac{t}{x} = F(C).$$

If $F(C) \neq 0$ and $t \neq 0$, it follows that for large x , $\frac{x}{t} \sim \frac{1}{F(C)}$. Since $n_t = n_1 + \sum_{i=1}^{t-1} (n_{i+1} - n_i)$ and $n_t \sim x$, one can conclude that the gaps in (2) are bounded in average.

A well-known theorem by ERDŐS and WINTNER gives the following criterion to determine if an additive function has a limiting distribution.

Theorem 1. *In order that an additive function $f(n)$ should possess a limiting distribution, it is both necessary and sufficient that the three series*

$$(3) \quad \sum_{|f(p)| > 1} \frac{1}{p}, \quad \sum_{|f(p)| \leq 1} \frac{f(p)}{p} \quad \text{and} \quad \sum_{|f(p)| \leq 1} \frac{f^2(p)}{p}$$

converge.

One can use theorem 1 to prove the following lemma.

Lemma. *If $f(n)$ is a nonnegative, additive function which tends monotonically to 0 on the sequence of primes, and which satisfies (1), then $f(n)$ has a limiting distribution.*

PROOF. The first series in (3) is a finite sum since $f(p)$ tends to 0 monotonically.

The convergence of the third series in (3) will be a consequence of the convergence of the middle series and the fact that $f(p) \geq 0$.

The convergence of the middle series in (3) follows from the assumption (1), which implies that there exists $N > 0$ such that for all $n \geq N$

$$\sum_{n < p \leq 2n} f(p) \leq \frac{1}{2}.$$

Therefore

$$\sum_{|f(p)| \leq 1} \frac{f(p)}{p} = \sum_{\substack{|f(p)| \leq 1 \\ p \leq N}} \frac{f(p)}{p} + \sum_{i=0}^{\infty} \sum_{2^i N < p \leq 2^{i+1} N} \frac{f(p)}{p}.$$

The first sum is a finite sum and the double sum can be majorized by

$$\sum_{i=0}^{\infty} \frac{1}{2^i N} \sum_{2^i N < p \leq 2^{i+1} N} f(p) \leq \sum_{i=0}^{\infty} \frac{1}{2^i N} \frac{1}{2} = \frac{1}{N}.$$

Therefore each series in (3) converges and the lemma has been proved.

Application of the lemma therefore implies that the gaps $(n_{i+1} - n_i)$ are bounded in average. But this does not exclude the possibility that $\limsup_{i \rightarrow \infty} (n_{i+1} - n_i) = \infty$, which in fact is often the case. But it is possible, and this is the purpose of the present note, to show that there is a function $k(x)$ such that $\limsup_{x \rightarrow \infty} \frac{n_{i+1} - n_i}{k(x)} \leq 1$. The function $k(x)$ will be determined through the method of proof which Kátai developed for his minimax theorem. Let us first state the result of KÁTAI [2].

Theorem 2. *Let $f(n)$ be a nonnegative additive function, which tends monotonically to 0 on the sequence of primes. Let*

$$\psi(y) = \sum_{p \leq y} f(p)$$

$$\beta(p) = \sup_{\alpha \geq 1} f(p^\alpha)$$

For $k \geq 1$, let $E_k(x) = \max_{n \leq x} \min \{f(n+1), f(n+2), \dots, f(n+k)\}$. Assume that

$$\psi(2y) - \psi(y) = o(1), \quad y \rightarrow \infty$$

$$\psi(y) \rightarrow \infty$$

$$\sum_p |\beta(p) - f(p)| < \infty$$

Then

$$\lim_{x \rightarrow \infty} \left\{ E_k(x) - \frac{\psi(\log x)}{k} \right\} = B_k + C_k - \frac{\psi(k)}{k}$$

where $B_k = \sup_{n \geq 1} \frac{1}{k} \sum_{j=1}^k \sum_{\substack{p \leq k \\ p^\alpha \parallel n+j}} f(p^\alpha)$ and $C_k = \frac{1}{k} \sum_{p \leq k} (\beta(p) - f(p))$.

In the proof of theorem 2, k is considered fixed. But as long as $k(x) < \log x$, one can let $k = k(x)$ depend upon x without altering the proof. We now prove the following result:

Theorem 3. Let $f(n)$ be a nonnegative, strongly additive function which tends monotonically to 0 on the sequence of primes. Let $A = \sum_p \frac{f(p)}{p}$ and let $C > A$ determine a sequence of integers $n_1 < \dots < n_i < \dots$ by the condition $f(n_i) \leq C$. Let

$$k(x) = \frac{1}{C-A} \sum_{p \leq \log x} f(p).$$

Then
$$\limsup_{x \rightarrow \infty} \frac{(n_{i+1} - n_i)}{k(x)} \leq 1.$$

PROOF. During the proof of the minimax theorem KÁTAI shows that for fixed $k \geq 1$

$$E_k(x) \leq B_k + C_k + \frac{\psi(\log x)}{k} - \frac{\psi(k)}{k} + (\psi(4 \log x) - \psi(\log x)).$$

For a strongly additive function $C_k = 0$, and

$$B_k = \frac{1}{k} \sum_{j=1}^k \sum_{\substack{p \leq k \\ p \mid n+j}} f(p) \leq \frac{1}{k} \sum_{\substack{p \leq k \\ p \mid \prod_{j=1}^k n+j}} f(p) \left\{ \left\lfloor \frac{k}{p} \right\rfloor + 1 \right\}.$$

If one drops the divisibility requirement and replaces $\left\lfloor \frac{k}{p} \right\rfloor$ by $\frac{k}{p}$ one obtains the further inequality

$$B_k < \frac{1}{k} \left\{ \sum_{p \leq k} f(p) \frac{k}{p} + \sum_{p \leq k} f(p) \right\} = \sum_{p \leq k} \frac{f(p)}{p} + \frac{\psi(k)}{k}.$$

This implies that

$$E_k(x) < \frac{\psi(\log x)}{k} + \sum_{p \leq k} \frac{f(p)}{p} + o(1).$$

Although the inequalities were obtained under the assumption that k is fixed, they are equally valid if k depends upon x as long as $k(x) < \log x$. This will be true for large x if we define

$$k(x) = \frac{1}{C-A} \sum_{p \leq \log x} f(p) = \frac{1}{C-A} \psi(\log x).$$

For this $k(x)$, $E_k(x) < C + o(1)$ and $\limsup_{x \rightarrow \infty} E_k(x) \leq C$.

But if $E_k(x) < C$, then every interval of length $k(x)$ contains at least one n_i such that $f(n_i) \leq C$. Therefore, for all $n_i \leq x$, $(n_{i+1} - n_i) \leq k(x)$ which finishes the proof.

A similar result has been obtained by GALAMBOS [1] for the case where $f(p) = \frac{1}{p}$ and $C = 2$. In this case A can be shown to be less than 1 and $k(x) = \log \log \log x$.

References

- [1] J. GALAMBOS, On a conjecture of Kátaï concerning weakly composite numbers. *Proc. Amer. Math. Soc.* **96** (1986), 215—216.
- [2] I. KÁTAI, A minimax theorem for additive functions. *Publ. Math. (Debrecen)* **30** (1983), 249—252

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