N-recurrent V-manifolds and Takeno's conjecture

By F. SÖLER (Moncton)

1. Introduction

TAKENO and KITAMURA [1] have introduced the V-manifolds and they have developed the theory of V-space-times (cf. references in [2]). Later TAKENO [2] has studied the recurrency conditions for V-manifolds. He found the same conditions for 1, 2, and 3-recurrent V-manifolds, and he conjectures that this condition is also true for n-recurrent manifolds ($n \geqq 4$). In this paper, the author will formulate again the recurrency conditions in a some different way, to find among other theorems a proof of Takeno's conjecture. It will be shown here that the same results as in [2] can be found easily without computing $\nabla^2 R$ and $\nabla^3 R$ in a local chart. The computations for finding the components of different tensors were realized with an IBM-370 using methods developped by the author [3].

A V-manifold is defined here to have, in a local coordinate system, the following non-vanishing components for the covariant metric tensor:

(1-1) $$g_{11} = -1 \quad g_{22} = -B \quad g_{33} = -C \quad g_{44} = D$$

where B, C and D are functions of a single variable x, and are of class C^∞:

2. Recurrent and symmetric manifolds

Symmetric manifolds, in short s-manifolds were introduced and completely investigated by CARTAN [4]. Recurrent manifolds, in short r-manifolds, were introduced by RUSE [5] and most of their properties investigated by WALKER [6].

Let R be the Riemann—Christoffel tensor defined by: $R(x, y)z = [\nabla_x, \nabla_y]z - \nabla_{[x,y]}z$ where $x, y, z \in \mathfrak{R}$ are vector fields and ∇ is the connexion on the manifold. It will be supposed here that the torsion: $T(x, y) = \nabla_x y - \nabla_y x - [x, y] \equiv 0$, is zero.

The recurrency condition can be formulated:

(2-1) $$\nabla R = R \otimes \boldsymbol{a}$$

where $\boldsymbol{a}$ is a 1-form.

In a similar way an r^n-manifold is defined, when the following condition is satisfied:

(2-2) $$\nabla^{(n)} R = R \otimes \boldsymbol{t}$$

where $\boldsymbol{t} \in \overset{n}{\otimes} \mathfrak{R}^*$ is a tensor field of order n.

We will consider s-manifolds as special cases of r-manifolds; they arise if in (2-1) $\boldsymbol{a}\equiv 0$, i.e.: $\nabla R=0$; and we get an s^n-manifold if $\boldsymbol{t}\equiv 0$ in (2-1), i.e.: $\nabla^{(n)}R=0$.

The properties of r-manifolds have been investigated by several authors (see references in [7] and [8]). In this paper, we will use some of these properties which are well known. To simplify the presentation these will be given in form of propositions without proof.

Proposition 2-1. *Any r-manifold is a r^n-manifold i.e.: if* $\nabla R=R\otimes\boldsymbol{a}$ *then* $\nabla^{(n)}R=R\otimes\boldsymbol{t}$ *where* $\boldsymbol{t}=f(\boldsymbol{a}, \nabla\boldsymbol{a}, \ldots, \nabla^{(n-1)}\boldsymbol{a})$.

In particular for r^2-manifolds the recurrence tensor $\boldsymbol{b}$ is given by $\boldsymbol{b}=\boldsymbol{a}\otimes\boldsymbol{a}+\nabla\boldsymbol{a}$.

Proposition 2-2. If $\nabla R=0$ then $\nabla^{(n)}R=0$.

Proposition 2-3. *For r-manifolds the recurrence's 1-form is closed, i.e.* $\boldsymbol{a}=d\theta$*:*

Proposition 2-4. *If* $\boldsymbol{R}\neq 0$ *(scalar curvature) then the 1-form of recurrence is given by* $\boldsymbol{a}=d\ln\boldsymbol{R}$.

Proposition 2-5. *If* $\boldsymbol{R}\neq 0$ *then any r^2-manifold is necessarily an r-manifold.*

Proposition 2-6. *For non-simple r-manifolds the recurrence 1-form a is isotropic and recurrent. i.e.:* $\nabla\boldsymbol{a}=\tau\cdot\boldsymbol{a}\otimes\boldsymbol{a}$, *and* $g(\boldsymbol{a},\boldsymbol{a})=0$.

Proposition 2-7. *For r^2-manifolds the recurrence tensor is symmetric,* $\boldsymbol{A}\boldsymbol{b}=0$.

Let us now investigate the conditions for an r-manifold, with $\boldsymbol{a}\neq 0$, to be an s^2-manifold.

We must have $\boldsymbol{b}=0\Leftrightarrow\nabla\boldsymbol{a}=\tau\cdot\boldsymbol{a}\otimes\boldsymbol{a}$ and we can state:

Proposition 2-8. *The only r-manifolds which can be s^2-manifolds are the non-simple ones in Walker's classification* [6].

The properties of r-vector fields can be found elsewhere [9].

Proposition 2-9. *A manifold with constant curvature is necessarily an s-manifold.*

Proposition 2-10. *In an r-manifold there exists a tensor* $S\in\mathfrak{T}$ *which has the same properties of symmetry as R and such that it is parallel* $\nabla S=0$ *and is collinear to R; i.e.:* $S=\varphi R$ *and* φ *is given by:* $\varphi=\exp\{k\theta\}$ *where* $d\theta=\boldsymbol{a}$, *and if in particular* $\boldsymbol{R}\neq 0$ *then* $\varphi=\tau\cdot\boldsymbol{R}$.

PROOF. Since $\nabla S=\nabla\varphi\otimes R+\varphi\cdot\nabla R$ and $\nabla R=R\otimes\boldsymbol{a}$, we have:

$$\nabla\varphi+\varphi\cdot\boldsymbol{a}=0\Rightarrow\boldsymbol{a}=d\ln\varphi \tag{2-3}$$

Now as $\boldsymbol{a}=d\theta$ we get: $\varphi=e^{k\theta}$, and we have the first part of proposition 10.

The second part follows immediately from proposition 4. Because $\boldsymbol{a}=d\ln\boldsymbol{R}$, from (2-3) we get $d\ln\varphi=d\ln\boldsymbol{R}$.

3. Recurrency conditions for V-manifolds

In a coordinate system the recurrency condition can be written, as follows:

$$\nabla_\mu R_{\alpha\beta\varrho\sigma} = \boldsymbol{a}_\mu R_{\alpha\beta\varrho\sigma} \tag{3-1}$$

Proposition 3-1. *For indicies α, β, ϱ, σ and μ fixed, we have:*

i) *if* $R_{\alpha\beta\varrho\sigma} = 0$ *then* $\nabla_\mu R_{\alpha\beta\varrho\sigma} = 0$

ii) *if* $\nabla_\mu R_{\alpha\beta\varrho\sigma} = 0$ *and* $R_{\alpha\beta\varrho\sigma} \neq 0$ *then* $\boldsymbol{a}_\mu = 0$.

Let $X(x)$ be an arbitrary non-vanishing function of x. We put:

$$\varphi_n(X) \equiv X^{(n+1)}/X^{(n)} \tag{3-2}$$

for $n \in Z^+$ and $X^{(n)} \equiv \dfrac{d^n}{dx^n} X(x)$, $X^{(0)} \equiv X(x)$.

By straightforward computation it will be found that:

$$\frac{d}{dx}\varphi_n = \varphi_n(\varphi_{n+1} - \varphi_n) \quad \therefore \quad n \in Z^+ \tag{3-3}$$

We define the following functionals:

$$\begin{aligned} F(X) &\equiv \frac{1}{4}\varphi_0(X)[\varphi_0(X) - 2\varphi_1(X)] \\ H(X) &\equiv \frac{1}{2}\varphi_0(X)[\varphi_0^2(X) - 2\varphi_1(X)\varphi_0(X) + \varphi_1(X)\varphi_2(X)] \\ G(X, Y) &\equiv \frac{1}{4}\varphi_0(X)\cdot\varphi_0(Y) \\ J(X, Y) &\equiv \frac{1}{2}[\varphi_0(X) + \varphi_0(Y) - 2\varphi_1(X)]. \end{aligned} \tag{3-4}$$

We can now write the non-vanishing independents components of R:

$$\begin{aligned} R_{1212} &= -B\cdot F(B) \\ R_{1414} &= D\cdot F(D) \\ R_{2323} &= B\cdot C\cdot G(B, C) \\ R_{2424} &= -B\cdot D\cdot G(B, D) \\ R_{3131} &= -C\cdot F(C) \\ R_{3434} &= C\cdot D\cdot G(C, D) \end{aligned} \tag{3-5}$$

For the components $\nabla_1 R$ we have:

$$
(3\text{-}6)\qquad
\begin{aligned}
R_{1212/1} &= B\cdot H(B)\\
R_{1414/1} &= -D\cdot H(D)\\
R_{2323/1} &= -B\cdot C\cdot G(B,C)\ [J(B,C)+J(C,B)]\\
R_{2424/1} &= B\cdot D\cdot G(B,D)[J(B,D)+J(D,B)]\\
R_{3131/1} &= C\cdot H\cdot (C)\\
R_{3434/1} &= D\cdot C\cdot G(C,D)\cdot [J(C,D)+J(D,C)]
\end{aligned}
$$

and for the other components:

$$
(3\text{-}7)\qquad
\begin{aligned}
R_{1223/3} &= B\cdot C\cdot G(B,C)\cdot J(B,C)\\
R_{1224/4} &= -B\cdot D\cdot G(B,D)\cdot J(B,D)\\
R_{1424/2} &= B\cdot D\cdot G(B,D)\cdot J(D,B)\\
R_{1434/3} &= D\cdot C\cdot G(C,D)\cdot J(D,C)\\
R_{3123/2} &= C\cdot B\cdot G(C,B)\cdot J(C,B)\\
R_{3134/4} &= C\cdot D\cdot G(D,C)\cdot J(C,D)
\end{aligned}
$$

and the scalar curvature is given by:

$$
(3\text{-}8)\qquad \boldsymbol{R} = -2[F(B)+F(C)+F(D)-G(B,C)-G(B,D)-G(D,C)]
$$

By inspection of (3-5), (3-6) and (3-7), we get for example:

$$
R_{1212} \neq 0 \quad\text{and}\quad R_{1212/\alpha} = 0 \quad\text{for}\quad \alpha = 2, 3, 4
$$

and from proposition (3-1) ii) we get:

$$
\boldsymbol{a}_2 = \boldsymbol{a}_3 = \boldsymbol{a}_4 = 0.
$$

Now we can state

Proposition 3-2. *For V-manifolds the most general form for the recurrency's 1-form is given in a local chart by:* $\boldsymbol{a}_\mu = \delta^1_\mu \boldsymbol{a}_1$.

Let us consider now the different cases of V-manifolds when all of the functions B, C and D are not constants.

All of the $\bar{B}$, $\bar{C}$ and $\bar{D}$ $\left(\text{here } \bar{X} \equiv \dfrac{dX(x)}{dx}\right)$ cannot vanish, otherwise the manifold is flat.

Type I: All functions differents from zero $\bar{B}\cdot\bar{C}\cdot\bar{D}\neq 0$.

Type II: At most one vanishing function. Subcases:

$$
\begin{aligned}
II_a&:\ \bar{B}=0,\quad \bar{C}\cdot\bar{D}\neq 0\\
II_b&:\ \bar{C}=0,\quad \bar{B}\cdot\bar{D}\neq 0\\
II_c&:\ \bar{D}=0,\quad \bar{B}\cdot\bar{C}\neq 0
\end{aligned}
$$

Type III: At most one non-vanishing function Subcases:

$$III_a:\ \bar{B}\neq 0,\quad \bar{C}=\bar{D}=0$$
$$III_b:\ \bar{C}\neq 0,\quad \bar{B}=\bar{D}=0$$
$$III_c:\ \bar{D}\neq 0,\quad \bar{B}=\bar{C}=0$$

These three different types will be investigated in the following three sections.

4. Recurrency conditions and solutions for type I

We have $\bar{B}\cdot\bar{C}\cdot\bar{D}\neq 0$ and from (3-1), (3-5) and (3-6) we get:

A1) $$\boldsymbol{a}_1=-\frac{H(X)}{F(X)}=J(X,Y)+J(Y,X)$$

for X any of B, C, or D, and $X\neq Y$. From (3-7) and proposition (3-1) i) we get:

B1) $$J(X,Y)=0$$

for X, Y any B, C, or D, and $X\neq Y$. When B1 is satisfy follows that

i) $\boldsymbol{a}_1=0$

ii) $H(X)=0$, for X any B, C or D.

Let us investigated the case when condition B1 is satisfied. It follows from the definition of J in (3-4) that:

(4-2) $$J(X,Y)=0\Leftrightarrow \varphi_0(X)+\varphi_0(Y)=2\varphi_1(X)$$

and since (4-2) must be true when (X, Y) is replaced by any of the couples (B, C), (B, D), (C, B), (C, D), (D, B) or (D, C), we must have

(4-3) $$\varphi_0(X)=\varphi_0(Y)$$

for any X and Y. Then the equation (4-2) become

(4-4) $$\varphi_0(X)=\varphi_1(X)$$

and with the help of definition (3-2) the solution of this differential functional equation, can be found easily to be:

(4-5) $$X(x)=x\cdot e^{kx}$$

Now condition ii) of (4-1) imposes more restrictions on (4-5).

(4-6) $$\text{If}\quad H(X)=0\quad\text{then}\quad \varphi_0^2(X)-2\varphi_0(X)\varphi_1(X)+\varphi_1(X)=0$$

and with (4-4), (4-6) become

(4-7) $$\varphi_0(X)=\varphi_2(X)$$

From (3-2) we have $\varphi_n(X)=k$ and then from (3-4) and conditions (4-4) and (4-7) we get: $F(X)=-G(X, Y)=-\,k^2/4$.

The scalar curvature can be written:

$$R = -\varphi_0(B) \cdot J(B, C) - \varphi_0(C) \cdot J(C, D) - \varphi_0(D) \cdot J(D, B) \tag{4-8}$$

from B1 we get $R=0$.

We can formulate these results in the four following propositions.

Proposition 4-1. *Recurrent V-manifolds of type I impose the following restrictions on the functions:*

$$B = x_1 e^{kx}, \quad C = x_2 e^{kx} \quad \text{and} \quad D = x_3 e^{kx}$$

Proposition 4-2. *Recurrent V-manifolds of type I are conformal to flat manifold. i.e.: $W=0$ (Weyl's tensor).*

Proposition 4-3. *Recurrent V-manifolds of type I are manifolds of constant curvature. i.e. they verifying*

$$R_{\alpha\beta\varrho\sigma} = k(g_{\alpha\varrho} g_{\beta\sigma} - g_{\alpha\sigma} g_{\beta\varrho}).$$

Proposition 4-4. *Recurrent V-manifolds of type I are s-manifolds i.e. the recurrency's 1-form is $a \neq 0$.*

The last proposition follows also from proposition (4-3) and (2.9).

5. *Recurrency conditions for type II.* The solutions are similar for each one of the different subcases. For each subcase we have only one function constant.

From (3-1), (3-5) and (3-6), we get:

$$\text{A2)} \qquad a_1 = -\frac{H(X)}{F(X)} = -\frac{H(Y)}{F(Y)} = J(X, Y) + J(Y, X)$$

and from (3-7) and proposition 3-1:

$$\text{B2)} \qquad J(X, Y) = J(Y, X) = 0$$

here X and Y are different and they replace for each case II_a, II_b, II_c, and (C, D), (B, D), (B, C) respectively.

When B2 is satisfied we get·

i) $a_1 = 0$

(5-1)

ii) $H(X) = H(Y) = 0$.

Using definition (3-2) and with the help of (3-3) it is a straightforward matter to verify that if

$$\frac{d}{dx} F = -H \tag{5-2}$$

then condition (5-1) ii) is equivalent to:

$$F = C^{te} \neq 0. \tag{5-3}$$

Otherwise $X=(x+b)^2$, $Y=C^{te}$ i.e. the manifold is flat $R=0$.

The integral of the differential equation (5-3) can easily be found if we note that $\varphi_0^1=\varphi_0\cdot\varphi_1-\varphi_0^2$, and if we replace by (5-3) and put the constant $-\frac{1}{4}k^2$ we get the differential equation:

$$\frac{d\varphi_0}{k^2-\varphi_0^2}=\frac{1}{2}\,dx.$$

The solutions are

$$\begin{aligned}
&\text{i) } \varphi_0^2<k^2,\quad k^2>0,\quad X(x)=\cos h^2 z\\
&\text{ii) } \varphi_0^2<k^2,\quad k^2<0\Rightarrow X(x)=\cos^2 z\\
(5\text{-}4)\qquad &\text{iii) } \varphi_0^2>k^2,\quad k^2>0\Rightarrow X(x)=\sin h^2 z\\
&\text{iv) } \varphi_0^2>k^2,\quad k^2<0\Rightarrow X(x)=\sin^2 z\\
&\text{v) } k^2=0\Rightarrow X(x)=(x+b)^2
\end{aligned}$$

where $z=\frac{1}{2}kx+c$, k and c are constants.

If we replace the solution $X(x)$ in the differential equation $J(X,Y)=0$ we get: $\varphi_0(Y)=2\varphi_1(X)-\varphi_0(X)$, and the solutions for the above five cases are respectively:

i) $Y(x)=\sin h^2 z$
ii) $Y(x)=\sin^2 z$
iii) $Y(x)=\cos h^2 z$
iv) $Y(x)=\cos^2 z$
v) $Y=C^{te}$.

In each one of the four cases, the conditions B2 and (5-1) are identically satisfied.

Proposition 5-1. *Recurrent V-manifold of type* II *impose the followin restriction on the functions:*

a) $B=X,\ C=Y$ *and* $D=C^{te}$
b) $B=Y,\ C=X$ *and* $D=C^{te}$

with $X=\cos h^2(k_1x+k_2)$ *and* $Y=\sin h^2(k_1x+k_2)$ *where* k_1 *is a complex number with either the real or imaginary part vanishing.*

Proposition 5-2. *Recurrent V-manifolds of type* II *are s-manifolds.*

6. Recurrency conditions for type III.

In these three cases we have only one non-constant component for the covariant metric tensor. We get only one condition, namely:

$$\text{A3)}\qquad a_1=-\frac{H(X)}{F(X)}$$

All other components of R and ∇R are identically vanishing.

With (5-3) condition A3 can be written:

$$a_1 = \varphi_0(F)$$

By proposition 2-4 we have the equivalent condition:

$$a_1 = \partial_1 \ln R$$

where

$$R = -\frac{1}{2}\varphi_0(\varphi_0 - 2\varphi_1)$$

$$= -2F$$

Only when $F = C^{te} \neq 0$ is the manifold an s-manifold. This differential equation was solutioned in section 5, and the solution is given by (5-4).
When $F=0$ the manifold is flat; then $X(x)=(x+b)^2$.

Proposition 6-1. *V-manifolds of type* III *are always r-manifolds.*
In particular we have:

Proposition 6-2. *V-manifolds of* III *with $F=C^{te}$ are necessarily s-manifolds and the restrictions are: one of B, C and D is equal to X and the other are constants. Here $X(x)=\cos h^2 z$ or $X(x)=\sin h^2 z$.*

Proposition 6-3. *When $F \neq C^{te}$ then the scalar curvature is non-constant and the manifold is properly ($a \neq 0$) an r-manifold. The recurrency 1-form is given in the local chart by:*

$$a_\alpha = \delta^1_\alpha \partial_1 R.$$

7. Takeno's conjecture

The results in the last three sections can be summarized by the table:

Type		Conditions on B, C and D		Solution	Recurrency 1-form	Manifold
I		all non-constants		$X=xe^{kx}$	$a=0$	s-manifold
II		only one constant		$X=\cos h^2(kx+c)$ $Y=\sin h^2(kx+c)$	$a=0$	s-manifold
III	s	only one	$F=C^{te}\neq 0$	$X=\cos h^2(kx+c)$ or $X=\sin^2 h(kx+c)$	$a=0$	s-manifold
	r	non-constant	$F\neq C^{te}$	any other that $X\neq C^{te}$, $\cos h^2 z$, $\sin h^2 z$ or e^{kx}	$a_\alpha=\delta^1_\alpha\partial_1 R$	r-manifold

For type I, II and III_s we can apply propositions 2-2. They are always s^n-manifolds.
For type III_r we can apply propositions 2-1, 2-3, 2-4 and 2-5.

Let us investigate when a III_r-manifold is an s^2-manifold. The square of the norm of the 1-form is $g(\boldsymbol{a}, \boldsymbol{a}) = \boldsymbol{a}_1^2$ i.e.: it is always different from zero. Now we can apply propositions 2-6, 2-7, 2-8, 2-9 and 2-10.

We can state the following theorem:

Theorem 7-1. *Non-symmetric r^n-manifolds are necessarily of type* III_r, *i.e.: $F \neq C^{te}$, they are always non-symmetric r-manifolds.*

The following corollary asserts that Takeno's conjecture. cf. [2] is true.

Corollary 7-2. *For V-manifolds the conditions for r^n-manifolds are the same for any $n \geqq 1$.*

In particular Takeno's theorems cf. [2], [5-1], [5-2], [7-3] and [7-4] are here obtained without computing $\nabla^2 R$ and $\nabla^3 R$ and they can be generalized for any $n \geqq 1$.

References

[1] H. Takeno and S. Kitamura, On some special kind of spacetimes I. *Tensor, N. S.* **24** (1972), 266—272.

[2] H. Takeno, On the recurrency of the space-time V. *Tensor, N. S.* **27** (1973), 309—318.

[3] F. Söler. Non-numerical computer methods for relativists *Publ. Rapport Research M A* 82—02, *RMC* 1983.

[4] M. E. Cartan, Sur une classe remarquable d'espaces de Riemann. *Bull. Soc. Math. France* **54** (1926), 214—264.

[5] H. S. Ruse, On simply harmonic spaces *J. London Math. Soc.* **21** (1946), 243—247.

[6] A. G. Walker, On Ruses's spaces of recurrent curvature. *Proc. London Math. Soc.* **2**. Ser. **52** (1950), 36—64.

[7] F. Söler, Sur les variétés récurrentes conformément planes. *C. R. Acad. Sc. Paris, A,* **277** (1973), 601—603.

[8] C. D. Collinson and F. Söler, Second order conformally recurrent and birecurrent plane fronted waves. *Tensor, N. S.* **27** (1973), 37—40.

[9] F. Söler, *r*-vector fields on metric manifolds *Ann. Math. P. and Appl., 4,* **119** (1979), 1—8.

DÉPARTMENT DE MATHÉMATIQUE
UNIVERSITÉ DE MONCTON,
MONCTON, N.-B., CANADA

(Received May 14, 1985; in revised form February 13, 1986.)