

Remarks on generalized homogeneous deviation means

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1. Introduction. In a previous paper DARÓCZY and PÁLES [1] introduced the following concept:

A mean M defined on the set of positive numbers is called k -homogeneous if

$$(1) \quad M_{nk}(t \circ x) = M_k(t) M_n(x)$$

holds for all

$$n \in \mathbf{N}, x = (x_1, \dots, x_n) \in \mathbf{R}_+^n, t = (t_1, \dots, t_k) \in \mathbf{R}_+^k$$

where

$$t \circ x = (t_1 x_1, \dots, t_1 x_n, \dots, t_k x_1, \dots, t_k x_n) \in \mathbf{R}_+^{nk}.$$

It is easy to see that 1-homogeneous means are homogeneous in the usual sense.

The problem investigated in [1] was to find all the k -homogeneous deviation means if $k \geq 2$. A mean M on $\mathbf{R}_+$ is called a *deviation mean* if there exists a *deviation* (i.e. a function $E: \mathbf{R}_+ \times \mathbf{R}_+ \rightarrow \mathbf{R}$ with the properties

E1) $y \rightarrow E(x, y)$ is strictly decreasing for each fixed $x \in \mathbf{R}_+$;

E2) $E(x, x) = 0$ for all $x \in \mathbf{R}_+$)

such that $y = M_n(x_1, \dots, x_n)$ is the unique solution of the equation

$$E(x_1, y) + \dots + E(x_n, y) = 0$$

for all $n \in \mathbf{N}$, $x_1, \dots, x_n \in \mathbf{R}_+$. In this case we say that M is generated by E and we write $M = \mathfrak{M}_E$.

In [1] the following result was proved:

Theorem 1. If E is a differentiable deviation on $\mathbf{R}_+$, that is

$$E_2(x, y) = \frac{\partial}{\partial y} E(x, y)$$

exists for each $x, y \in \mathbf{R}_+$ and it is negative then $M = \mathfrak{M}_E$ is k -homogeneous for some fixed $k \geq 2$ if and only if there exist a multiplicative function $m: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ (i.e. $m(xy) = m(x)m(y)$ for $x, y \in \mathbf{R}_+$) and a positive constant a such that either

$$(2) \quad M_n(x_1, \dots, x_n) = \exp \left(\frac{\sum_{i=1}^n m(x_i) \ln x_i}{\sum_{i=1}^n m(x_i)} \right)$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}_+$, or

$$(3) \quad M_n(x_1, \dots, x_n) = \left(\sum_{i=1}^n m(x_i) x_i^a / \sum_{i=1}^n m(x_i) \right)^{1/a}$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}_+$.

The aim of the present note is to prove this theorem without assuming the differentiability of E .

2. Preliminary results. Let E be a fixed deviation on $\mathbb{R}_+$ throughout this paper and denote

$$(4) \quad f(x) = E(x, 1)$$

$$(5) \quad \mu(x) = \mathfrak{M}_{k,E}(x, \underbrace{1, \dots, 1}_{k-1})$$

for $x \in \mathbb{R}_+$.

Using these notations the following result can be verified (see [1] for the details):

Theorem 2. *If $M = \mathfrak{M}_E$ is a k -homogeneous mean for some fixed $k \geq 2$ then there exists a function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$(6) \quad E(x, y) = h(y)f(x/y)$$

for $x, y \in \mathbb{R}_+$, further, the functions f and μ satisfy the functional equation

$$(7) \quad \frac{f(xy)}{f(x)f(y)} + \frac{k-1}{f(x)} = \frac{f(\mu(x)y)}{f(\mu(x))f(y)}$$

for all $x, y \in \mathbb{R}_+ \setminus \{1\}$.

Based on this assertion the following theorem was proved in [1, Th. 2].

Theorem 3. *If E is a differentiable deviation on $\mathbb{R}_+$ and $M = \mathfrak{M}_E$ is a k -homogeneous mean ($k \geq 2$ is fixed), then f satisfies the functional equation*

$$(8) \quad 2 \frac{f(xy)}{f(x)f(y)} = \frac{f(x^2)}{f^2(x)} + \frac{f(y^2)}{f^2(y)}$$

for any $x, y \in \mathbb{R}_+ \setminus \{1\}$.

The proof of this theorem was the only point in [1] where the differentiability of E was used. Therefore, if we can deduce functional equation (8) from Theorem 2 without assuming E to be differentiable then we verify Theorem 1 for all deviations.

3. Main results. It follows from the definition of deviations and deviation means that

$$(9) \quad \text{sign } f(x) = \text{sign } (\mu(x) - 1) = \text{sign } (x - \mu(x)) = \text{sign } (x - 1)$$

for all $x \in \mathbb{R}_+$.

If E generates a multiplicative mean then, by Theorem 2,

$$f(xy)/f(x) = E(y, 1/x)/E(1, 1/x) \quad (x, y \in \mathbb{R}_+, \quad x \neq 1).$$

Thus, by property E1) of deviations we can see that

$$(10) \quad x \rightarrow f(xy)/f(x)$$

is a continuous function on $\mathbf{R}_+ \setminus \{1\}$ for each fixed $y \in \mathbf{R}_+$.

Now we can state our main result:

Theorem 4. Let $f: \mathbf{R}_+ \rightarrow \mathbf{R}$ and $\mu: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be arbitrary functions. Assume that

(i) (9) is satisfied if $x \in \mathbf{R}_+$;

(ii) (10) is a continuous function on $\mathbf{R}_+ \setminus \{1\}$ for all fixed $y \in \mathbf{R}_+$;

(iii) (7) holds for all $x, y \in \mathbf{R}_+ \setminus \{1\}$.

Then f satisfies functional equation (8) for all $x, y \in \mathbf{R}_+ \setminus \{1\}$.

PROOF. Let $y, z \in \mathbf{R}_+ \setminus \{1\}$ be fixed and define the function $G = G_{y,z}$ in the following way

$$G(x) = G_{y,z}(x) = \frac{f(xy)}{f(x)f(y)} - \frac{f(xz)}{f(x)f(z)}$$

for $x \in \mathbf{R}_+ \setminus \{1\}$.

We are going to show that G is the constant function. First we deduce two identities for G . Put z into (7) instead of y and subtract the resulting equation from (7). Then we get

$$(11) \quad G(x) = G(\mu(x))$$

for $x \in \mathbf{R}_+ \setminus \{1\}$.

Let $t \in \mathbf{R}_+$ with $1 \neq t \neq 1/\mu(x)$ and put $y = yt$ and $y = t$ into (7) to obtain

$$f(\mu(x))[f(xty) + (k-1)f(ty)] = f(x)f(\mu(x)ty)$$

and

$$(12) \quad f(x)f(\mu(x)t) = f(\mu(x))[f(xt) + (k-1)f(t)].$$

Multiplying these inequalities and then dividing by $f(x)f(\mu(x)) \neq 0$, we have

$$(13) \quad f(\mu(x)t)[f(xty) + (k-1)f(ty)] = f(\mu(x)ty)[f(xt) + (k-1)f(t)].$$

Since $f(x)f(\mu(x))f(\mu(x)t) \neq 0$, hence, by (12), $f(xt) + (k-1)f(t) \neq 0$. Therefore it follows from (13) that

$$(14) \quad \frac{f(xty)}{f(xt)f(y)} \frac{f(xt)}{f(xt) + (k-1)f(t)} + \frac{f(ty)}{f(t)f(y)} \frac{(k-1)f(t)}{f(xt) + (k-1)f(t)} = \frac{f(\mu(x)ty)}{f(\mu(x)t)f(y)}.$$

Putting $y = z$ into (14) and subtracting the resulting equation from (14) we obtain

$$(15) \quad G(xt) \frac{f(xt)}{f(xt) + (k-1)f(t)} + G(t) \frac{(k-1)f(t)}{f(xt) + (k-1)f(t)} = G(\mu(x)t).$$

Now let $x_0 > 1$ be fixed and define

$$H_{x_0} = \{x > 1 | G(x) = G(x_0)\}.$$

By our assumption (ii), G is a continuous function on $]1, \infty[$, therefore H_{x_0} is a closed set in the interval $]1, \infty[$.

Based on (15), we shall prove that H_{x_0} is convex. If H_{x_0} were not convex then (since it is closed) there exists $t, s \in H_{x_0}$, $t < s$ such that $r \notin H_{x_0}$ if $t < r < s$. Let $x = s/t$ in (15). Then $G(xt) = G(s) = G(t) = G(x_0)$ since $s, t \in H_{x_0}$. Therefore $G(\mu(x)t) = G(x_0)$, i.e.

$$r = \mu(x)t = \mu(s/t)t \in H_{x_0}.$$

However, by assumption (i),

$$t < \mu(s/t)t < s,$$

which is a contradiction.

The convexity of H_{x_0} means that it is a subinterval of $]1, \infty[$. If

$$\inf H_{x_0} = x^* > 1$$

then (since H_{x_0} is closed) $x^* \in H_{x_0}$. On the other hand, by (11), we have that $\mu(x^*)$ is also contained in H_{x_0} . But (i) implies that $1 < \mu(x^*) < x^*$ which contradicts $x^* > 1$. This contradiction proves that $x^* = 1$.

To prove that $G(x) = G(x_0)$ for all $x \in]1, \infty[$, write $x^* = \min(x_0, x)$. Now $x^* \in H_{x_0} \cap H_x$ since $]1, x_0] \subseteq H_{x_0}$, $]1, x] \subseteq H_x$. But then, by the definition of H_{x_0} and H_x , $G(x_0) = G(x^*) = G(x)$.

Therefore we have seen that $G(x) = c'$ for $x > 1$. A similar argument shows that $G(x) = c''$ for $0 < x < 1$ is also valid. We have only to verify that $c' = c''$. Let $x > 1$ and choose $t \in \mathbb{R}_+$ so that $t < 1$ and $tx, t\mu(x) > 1$. Then, applying (15), we obtain $c' = c''$.

Now we can show that (8) holds. Let $x, y \in \mathbb{R}_+ \setminus \{1\}$. As we have proved, $G = G_{x,y}$ is a constant function. Therefore

$$G_{x,y}(x) = G_{x,y}(y)$$

i.e. (8) is satisfied.

Thus the proof of the theorem is complete.

Based on the above result, we can restate Theorem 3 in the following form

Theorem 3*. *If E is a deviation on $\mathbb{R}_+$ and $M = \mathfrak{M}_E$ is a k -homogeneous mean ($k \geq 2$ is fixed) then $f(x) = E(x, 1)$ satisfies the functional equation (8) for all $x, y \in \mathbb{R}_+ \setminus \{1\}$.*

Using the same method that was followed in [1], Theorem 3 implies

Theorem 1*. *If M is a deviation mean then it is k -homogeneous for some fixed $k \geq 2$ if and only if there exist a multiplicative function $m: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and a positive, constant a such that either (2) or (3) is satisfied for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{R}_+$.*

Reference

- [1] Z. DARÓCZY—Zs. PÁLES, Generalized homogeneous deviation means, *Publ. Math. (Debrecen)* **33** (1986), 53—65.

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