

Prolongations of G -structures to the frame bundle of second order

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Introduction

Let M be an n -dimensional differentiable manifold, TM its tangent bundle $T^r M$ its tangent bundle of order r , $r \geq 2$, FM its frame bundle and $F^2 M$ its frame bundle of second order.

The theory of prolongations of G -structures on M to appropriate G -structures on TM and $T^r M$ has been introduced by A. MORIMOTO in [7] and [8], respectively. Recently, L. A. CORDERO and M. DE LEÓN [1] has developed a similar theory for FM .

In this paper, we extend this theory for $F^2 M$ and show, by studying the prolongations of some classical G -structures on M , how the different definitions of lifts given in [3], [4] and [13] fit nicely in this general framework.

In a forthcoming paper, we shall show that a similar theory can be developed for the diagonal lifts of tensor fields on M to $F^2 M$ introduced in [13].

§ 1. Preliminaires

1.1. The tangent bundle of n^2 -velocities.

Let M be an n -dimensional manifold. We denote by $T_n^2 M$ the set of all 2-jets at 0 of differentiable mappings $R^n \rightarrow M$. Let $\pi_n^2: T_n^2 M \rightarrow M$ be the target projection, i.e., $\pi_n^2(j_0^2 f) = f(0)$. In $T_n^2 M$ we consider the following manifold structure (see MORIMOTO [9]): Let N_2 denote the set of all n -tuples $v = (v_1, \dots, v_n)$ of non negative integers such that $|v| = v_1 + \dots + v_n \leq 2$. Every chart (U, x^i) on M induces a chart $((\pi_n^2)^{-1}(U) = T_n^2 U, x_{(v)}^i: i = 1, \dots, n; v \in N_2)$ on $T_n^2 M$, called the induced chart, where

$$x_{(v)}^i(j_0^2 f) = \frac{1}{v!} \left(\frac{\partial}{\partial t} \right)^v (x^i \circ f)_{t=0} = \frac{1}{v!} \frac{\partial^{v_1 + \dots + v_n}}{\partial t_1^{v_1} \dots \partial t_n^{v_n}} (x^i \circ f)_{t=0}$$

being $(t_1, \dots, t_n)$ the natural coordinates of R^n and $v! = v_1! \dots v_n!$.

The set N_2 has $1 + n + \frac{n(n+1)}{2}$ elements which can be ordered by the fol-

lowing bijection

$$N_2 \rightarrow \left\{ 0, 1, \dots, n, \dots, n + \frac{n(n+1)}{2} \right\}$$

$$\Psi(0, \dots, 0) = 0,$$

$$\Psi(0, \dots, 1, \dots, 0) = i \quad (\text{we place } 1 \text{ in } i\text{-th position})$$

$$\Psi(0, \dots, 1, \dots, 1, \dots, 0) = j + \frac{(2n-i)(i-1)}{2} + n,$$

$$(\text{we place } 1 \text{ in } i\text{-th and } j\text{-th positions, } i < j)$$

$$\Psi(0, \dots, 2, \dots, 0) = i + \frac{(2n-i)(i-1)}{2} + n,$$

$$(\text{we place } 2 \text{ in } i\text{-th position}).$$

If $v = (v_1, \dots, v_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ are two n -tuples, then we write $v \pm \lambda = (v_1 \pm \lambda_1, \dots, v_n \pm \lambda_n)$ and $v = \lambda$ if $v_i = \lambda_i$ for $i = 1, \dots, n$.

It is clear that $x_{(0, \dots, 0)}^i(j_0^2 f) = x^i(f(0))$ and we shall denote these coordinates $x_{(0, \dots, 0)}^i$ by x^i . In the sequel, we shall denote by $(x^i, x_{(\lambda)}^i, x_{(v)}^i)$ the induced coordinate system in $T_n^2 U$, where $\lambda, v \in N_2$ with $|\lambda| = 1, |v| = 2$.

Let (U, x^i) and $(\bar{U}, \bar{x}^i)$ be two intersecting charts on M and the coordinate transformation in $U \cap \bar{U}$ given by $\bar{x}^i = \bar{x}^i(x)$. Then the transformation of induced coordinates in $T_n^2 U \cap T_n^2 \bar{U} = T_n^2(U \cap \bar{U})$ is given by

$$(1.1) \quad \begin{aligned} \bar{x}^i &= x^i & \bar{x}_{(\lambda)}^i &= \frac{\partial \bar{x}^i}{\partial x^r} x_{(\lambda)}^r, \quad |\lambda| = 1, \\ \bar{x}_{(v)}^i &= \frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^s} \sum_{\substack{|\lambda|=1 \\ \lambda \in N_2}} x_{(v-\lambda)}^r x_{(\lambda)}^s + \frac{\partial \bar{x}^i}{\partial x^r} x_{(v)}^r, \quad |v| = 2. \end{aligned}$$

The Jacobian of (1.1) is given by the matrix

$$(1.2) \quad \begin{pmatrix} a & 0 & \dots & 0 & 0 \\ B_1 & a & & & \\ \vdots & 0 & \ddots & & \\ B_n & 0 & & a & \\ C_{n+1} & 0 & & 0 & a \\ \vdots & & & & & a \\ C_n + \frac{n(n+1)}{2} & (*) & & 0 & & 0 & a \end{pmatrix}$$

being (*) the matrix

$$\begin{pmatrix} B_1 & & & & & \\ \vdots & B_1 & & & & \\ \vdots & & \ddots & & & \\ \vdots & & & 0 & & \\ \vdots & & & & \ddots & \\ \vdots & & & & & 0 \\ B_n & 0 & & & & B_1 \\ 0 & B_2 & & & & \\ \vdots & & \ddots & & & \\ 0 & & & 0 & & \\ \vdots & & & & \ddots & \\ B_n & & & & & B_2 \end{pmatrix} \\ \hline \begin{pmatrix} & B_k & & & \\ & \vdots & & & \\ 0 & & \ddots & & 0 \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \\ & B_n & & & & B_k \end{pmatrix} \\ \hline \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ 0 & & B_{n-1} & & 0 & \\ & & B_n & & B_{n-1} & \\ & & 0 & & B_n & \end{pmatrix}$$

where

$$a = \left(\frac{\partial \bar{x}^i}{\partial x^t} \right)$$

$$B_k = \left(\frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^t} x_{\psi^{-1}(k)}^r \right), \quad k = 1, \dots, n,$$

$$C_{j+\frac{(2n-h)(h-1)}{2}+n} = \left(\frac{1}{2} \frac{\partial^3 \bar{x}^i}{\partial x^r \partial x^s \partial x^t} x_{\psi^{-1}(j)}^r x_{\psi^{-1}(h)}^s x_{\psi^{-1}(h)}^t + \right. \\ \left. + \frac{\partial^3 \bar{x}^i}{\partial x^r \partial x^t} x_{\psi^{-1}\left(j+\frac{(2n-h)(h-1)}{2}+n\right)}^r \right), \quad j \leq h,$$

and

$$x_{\psi^{-1}(j)}^r x_{\psi^{-1}(h)}^s = x_{\psi^{-1}(j)}^r x_{\psi^{-1}(h)}^s + x_{\psi^{-1}(h)}^r x_{\psi^{-1}(j)}^s$$

Definition 1.1. The differentiable manifold $T_n^2 M$ with projection π_n^2 is called the tangent bundle of n^2 -velocities of M .

Now, we shall recall some properties of the functor T_n^2 :

(A) Let $f: M \rightarrow N$ be a map of a manifold M into another manifold N . Then the map f induces a map $T_n^2 f: T_n^2 M \rightarrow T_n^2 N$ defined by $T_n^2 f(j_0^2 g) = j_0^2 (f \circ g)$. Moreover, if f has maximal rank, then $T_n^2 f$ has maximal rank. Therefore, if f is an immersion (resp. diffeomorphism) then $T_n^2 f$ is an immersion (resp. diffeomorphism), and if f is a diffeomorphism then $(T_n^2 f)^{-1} = T_n^2 (f^{-1})$.

(B) Let M and N be two arbitrary manifolds and $M \times N$ the manifold product. Then $T_n^2 (M \times N)$ and $T_n^2 M \times T_n^2 N$ are canonically diffeomorphic.

(C) Let G be a Lie group with multiplication $\mu: G \times G \rightarrow G$. Then $T_n^2 G$ is a Lie group with multiplication $T_n^2 \mu: T_n^2 G \times T_n^2 G \rightarrow T_n^2 G$. Moreover, if $h: G \rightarrow G'$ is a homomorphism of Lie groups then $T_n^2 h: T_n^2 G \rightarrow T_n^2 G'$ is so also. Therefore, if G is a Lie subgroup of G' , then $T_n^2 G$ is a Lie subgroup of $T_n^2 G'$.

(D) If a Lie group G operates on a manifold M differentiably and effectively then $T_n^2 G$ operates on $T_n^2 M$ differentiably and effectively.

(E) If $P(M, \pi, G)$ is a principal fibre bundle then $T_n^2 P(T_n^2 M, T_n^2 \pi, T_n^2 G)$ is a principal fibre bundle which be called the induced bundle. In fact, let U be a coordinate neighbourhood on M ; if $\varphi_U: U \times G \rightarrow \pi^{-1}(U)$ is the local trivialization of P then $T_n^2(\varphi_U): T_n^2 U \times T_n^2 G \rightarrow T_n^2(\pi^{-1}(U))$ is the local trivialization of $T_n^2 P$.

1.2. The frame bundle of second order.

Let M be an n -dimensional manifold. We denote by $F^2 M$ the set of all 2-jets at 0 of diffeomorphisms of open neighbourhoods of 0 in R^n onto open subsets of M . Let $\pi^2: F^2 M \rightarrow M$ be the target projection $\pi^2(j_0^2 f) = f(0)$. Then $\pi^2: F^2 M \rightarrow M$ is a principal fibre bundle with the structural group L_n^2 of all 2-jets with the source and the target at 0 of local diffeomorphisms of R^n . The group L_n^2 operates on $F^2 M$ on the right in the natural way $((j_0^2 f)(j_0^2 g) = j_0^2(f \circ g))$, where $j_0^2 f$ and $j_0^2 g$ belong to $F^2 M$ and L_n^2 , respectively.

Let us remark that $F^2 M$ is an open and dense subset of $T_n^2 M$. If $f: M \rightarrow N$ is a local diffeomorphism then we define its prolongation $f^2: F^2 M \rightarrow F^2 N$ by $f^2(j_0^2 g) = j_0^2(f \circ g)$, and it is clear that $T_n^2 f|_{F^2 M} = f^2$.

Every chart (U, x^i) on M induces a chart $((\pi^2)^{-1}(U) = F^2 U, x_{(v)}^i: i=1, \dots, n; v \in N_2)$ on $F^2 M$, where

$$x_{(v)}^i(j_0^2 f) = \frac{1}{v!} \left(\frac{\partial}{\partial t} \right)^v (x^i \circ f)_{t=0}.$$

We shall denote the induced coordinate system by $(x^i, x_{(\lambda)}^i, x_{(v)}^i)$, where $\lambda, v \in N_2$, with $|\lambda|=1, |v|=2$.

Now, we can consider the restriction to $F^2 M$ of the γ -lifts ($\gamma \in N_2$) of tensor fields on M to $T_n^2 M$ (see [3], [4]):

(A) Lifts of functions.

If f is a differentiable function on M , we define the 0-lift, (λ) -lift (v) -lift ($\lambda, v \in N_2: |\lambda|=1, |v|=2$) as the functions $f^0, f^{(\lambda)}, f^{(v)}$ on $F^2 M$ given by

$$(1.3) \quad \begin{aligned} f^0 &= f^V = (\pi^2)^* f, \\ f^{(\lambda)} &= x_{(\lambda)}^i \partial_i f, \\ f^{(v)} &= \sum_{\substack{\lambda \in N_2 \\ |\lambda|=1}} x_{(v-\lambda)}^i x_{(\lambda)}^j \partial_i \partial_j f + x_{(v)}^i \partial_i f, \end{aligned}$$

where $\partial_i f = \frac{\partial f}{\partial x^i}$.

It is convenient to define $f^{(v)}=0$ if $v \notin N_2$.

(B) *Lifts of vector fields.*

If X is a vector field on M , we define the complete lift (or 0-lift), (λ) -lift and (ν) -lift to F^2M as the vector fields on F^2M given by

$$(1.4) \quad \begin{aligned} X^C &= (X^i)^V \frac{\partial}{\partial x^i} + \sum_{\substack{\lambda \in N_2 \\ |\lambda|=1}} (X^i)^{(\lambda)} \frac{\partial}{\partial x_{(\lambda)}^i} + \sum_{\substack{\nu \in N_2 \\ |\nu|=2}} (X^i)^{(\nu)} \frac{\partial}{\partial x_{(\nu)}^i}, \\ X^{(\lambda)} &= (X^i)^V \frac{\partial}{\partial x_{(\lambda)}^i} + \sum_{\substack{\mu \in N_2 \\ |\mu|=1}} (X^i)^{(\mu)} \frac{\partial}{\partial x_{(\lambda+\mu)}^i}, \\ X^{(\nu)} &= (X^i)^V \frac{\partial}{\partial x_{(\nu)}^i}, \end{aligned}$$

where X^i are the local components of X .

It is convenient to define $X^{(\gamma)}=0$ if $\gamma \notin N_2$.

(C) *Lifts of 1-forms.*

Similarly, if Θ is an i -form on M , we can define the corresponding lifts Θ^0 , $\Theta^{(\lambda)}$, $\Theta^{(\nu)}$ to F^2M as the i -forms on F^2M given by

$$(1.5) \quad \begin{aligned} \Theta^0 &= \Theta^V = (\pi^2)^* \Theta, \\ \Theta^{(\lambda)} &= (\Theta_i) dx_{(\lambda)}^i + (\Theta_i)^V dx^i, \\ \Theta^{(\nu)} &= (\Theta_i)^V dx_{(\nu)}^i + \sum_{\substack{\mu \in N_2 \\ |\mu|=1}} (\Theta_i)^{(\nu-\mu)} dx_{(\mu)}^i + (\Theta_i)^V dx^i, \end{aligned}$$

where Θ^i are the local components of Θ .

It is convenient to define $\Theta^{(\lambda)}=0$, if $\lambda \notin N_2$.

From (1.3), (1.4) and (1.5), we deduce

Proposition 1.2. *Let X and Θ be a vector field and 1-form on M , respectively. Then we have*

$$(1) \quad \Theta^V(X^C) = (\Theta(X))^V, \quad \Theta^V(X^{(\lambda)}) = 0, \quad \Theta^V(X^{(\nu)}) = 0,$$

$$(2) \quad \Theta^{(\alpha)}(X^C) = (\Theta(X))^{(\alpha)}, \quad \Theta^{(\alpha)}(X^{(\lambda)}) = \delta^{\alpha\lambda}(\Theta(X))^V, \quad \Theta^{(\alpha)}(X^{(\nu)}) = 0,$$

$$(3) \quad \Theta^{(\beta)}(X^C) = (\Theta(X))^{(\beta)}, \quad \Theta^{(\beta)}(X^{(\lambda)}) = (\Theta(X))^{(\beta-\lambda)}, \quad \Theta^{(\beta)}(X^{(\nu)}) = \delta^{\beta\nu}(\Theta(X))^V,$$

where $\alpha, \beta, \lambda, \nu \in N_2$, with $|\alpha|=|\lambda|=1$, $|\beta|=|\nu|=2$. $\#$

(D) *Lifts of tensor fields of type (1, 1).*

If F is a tensor field of type (1, 1) on M , we define the (γ) -lift ($\gamma \in N_2$) of F to F^2M as the tensor field $F^{(\gamma)}$ of type (1, 1) on F^2M given by

$$F^{(\gamma)}(X^C) = (FX)^{(\gamma)}, \quad F^{(\gamma)}(X^{(\xi)}) = (FX)^{(\gamma-\xi)},$$

for any vector field X on M , where $\xi \in N_2$, $1 \leq |\xi| \leq 2$.

We shall denote $F^{(0, \dots, 0)}$ by F^C . F^C is called the complete lift of F to F^2M . In [3], Gancarzewicz proved that F^C is completely characterized by the identity $F^C(X^C) = (FX)^C$, for any vector field X on M .

(E) *Lifts of tensor fields of type (0, 2).*

Finally, let g be a tensor field of type (0, 2) on M . Then we define the (γ) -lift of g to F^2M , $\gamma \in N_2$, as the tensor field of type (0, 2) on F^2M given by

$$g^{(\gamma)}(X^{(\alpha)}, Y^{(\beta)}) = (g(X, Y))^{(\gamma - \alpha - \beta)},$$

for any vector field X on M , where $\alpha, \beta \in N_2$.

§ 2. Imbedding of $T_n^2 GL(n)$ into $Gl(N)$, $N = n \left(1 + n + \frac{n(n+1)}{2}\right)$.

Let R^n be the n -dimensional euclidean space, and consider the tangent bundle of n^2 -velocities $T_n^2 R^n$ of R^n . It is clear that $T_n^2 R^n$ is a vector space of dimension N . In fact, for any two 2-jets $j_0^2 f$ and $j_0^2 g$, we define their sum by: $j_0^2 f + j_0^2 g = j_0^2(f+g)$, where $(f+g)(t) = f(t) + g(t)$, for $t \in R^n$. For any $c \in R^n$ we define the scalar multiplication of $j_0^2 f$ by c as follows: $c \cdot (j_0^2 f) = j_0^2(c \cdot f)$, where $(c \cdot f)(t) = c \cdot f(t)$, for $t \in R^n$.

Let (t^i) be the natural coordinate system on R^n and let $(t_{(\alpha)}^i)$: $i = 1, \dots, n$; $\alpha \in N_2$ be the induced coordinate system on $T_n^2 R^n$. Then the sum and scalar multiplication in $T_n^2 R^n$ are as follows:

$$(t_{(\alpha)}^i) + (t'_{(\alpha)}^i) = (t_{(\alpha)}^i + t'_{(\alpha)}^i), \quad c \cdot (t_{(\alpha)}^i) = (c \cdot t_{(\alpha)}^i).$$

Let $Gl(n)$ be the general linear group. Then $T_0^2 Gl(n)$ is a Lie group. Let $\tilde{A} \in T_n^2 Gl(n)$ be the 2-jet defined by the map $A: R^n \rightarrow Gl(n)$, and $(A_i^h(t))$ the matrix that represents to $A(t)$ for each $t \in R^n$. The element $\tilde{A}$ can be identified to the $\left(1 + n + \frac{n(n+1)}{2}\right)$ -tupla

$$(a; \dots, B^{(\lambda)}, \dots; \dots, C^{(v)}, \dots), \quad |\lambda| = 1, |v| = 2,$$

where

$$a = (A_i^h(0)), \quad a \in Gl(n),$$

$$B^{(\lambda)} = \frac{1}{\lambda!} \left[\left(\frac{\partial}{\partial t} \right)^\lambda (A_i^h(t)) \right]_{t=0}, \quad B^{(\lambda)} \in gl(n),$$

$$C^{(v)} = \frac{1}{v!} \left[\left(\frac{\partial}{\partial t} \right)^v (A_i^h(t)) \right]_{t=0}, \quad C^{(v)} \in gl(n),$$

being $gl(n)$ the Lie algebra of $Gl(n)$.

Now, let $Gl(n) \times R^n \rightarrow R^n$ be the usual operation of the general linear group $Gl(n)$ on R^n and consider the induced operation

$$T_n^2 Gl(n) \times T_n^2 R^n \rightarrow T_n^2 R^n \\ (\tilde{A}, \tilde{P}) \rightarrow \tilde{A} \cdot \tilde{P}$$

If $\tilde{A} = j_0^2 A$, $\tilde{P} = j_0^2 P$, where $A: R^n \rightarrow Gl(n)$, $P: R^n \rightarrow R^n$ are defined by $A(t) = (A_i^h(t))$, $P(t) = (\xi^h(t))$, then $\tilde{A} \cdot \tilde{P} = j_0^2(A \cdot P)$, being $A \cdot P: R^n \rightarrow R^n$ the map defined by

$(A \cdot P)(t) = A(t) \cdot P(t)$ for $t \in R^n$. Therefore, if $(A \cdot P)(t) = (\xi^h(t))$, then $\xi^h(t) = A_i^h(t) \xi^i(t)$ for $t \in R^n$. Let $(t_{(a)})$ and $(t_{(a)}^i)$ the induced coordinates of $\tilde{P}$ and $\tilde{A} \cdot \tilde{P}$, respectively. Then we obtain

$$(2.1) \quad \begin{aligned} {}'t^h &= a_i^h t^i \\ {}'t_{(\lambda)}^h &= (B^{(\lambda)})_i^h t^i + a_i^h t_{(\lambda)}^i, \quad |\lambda| = 1, \\ {}'t_{(v)}^h &= (C^{(v)})_i^h t^i + \sum_{\substack{\lambda \in N_2 \\ |\lambda|=1}} (B^{(v-\lambda)})_i^h t_{(\lambda)}^i + a_i^h t_{(v)}^i, \quad |v| = 2, \end{aligned}$$

from where deduce that $T_n^2 Gl(n)$ operates on $T_n^2 R^n$ as a group of linear transformations. So, each element $\tilde{A} = j_0^2 A \in T_n^2 Gl(n)$ can be identified to a linear automorphism of $T_n^2 R^n \cong R^N$. Therefore, we deduce the following proposition.

Proposition 2.1. *There is a canonical homomorphism ϱ_n^2 of $T_n^2 Gl(n)$ into $Gl(N)$ which is given by the following equality:*

$$(2.2) \quad \varrho_n^2(\tilde{A}) = \begin{pmatrix} a & & & & \\ B_1 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & a & \ddots & & & \\ \vdots & 0 & \ddots & \ddots & & \\ B_n & \vdots & & & 0 & \\ C_{n+1} & 0 & \cdots & 0 & a & \cdots & 0 \\ \vdots & & & & \vdots & \ddots & \\ C_n + \frac{n(n+1)}{2} & & (*) & & 0 & \cdots & 0 & a \end{pmatrix}$$

where $(*)$ is the following matrix

$$\begin{pmatrix} B_1 & & & & \\ \vdots & B_1 & & & \\ & \ddots & \ddots & & 0 \\ & & 0 & \ddots & \\ & & & \ddots & B_1 \\ B_n & 0 & & & \\ 0 & B_2 & & & \\ & \vdots & \ddots & & 0 \\ 0 & & 0 & \ddots & \\ & & & \ddots & B_n & B_2 \\ \hline & & B_k & & & \\ & 0 & \vdots & \ddots & & 0 \\ & & \vdots & 0 & \ddots & \\ & & B_n & & B_k \\ \hline & & & B_{n-1} & 0 \\ 0 & & & B_n & B_{n-1} \\ & & & 0 & B_n \end{pmatrix}$$

being

$$B_i = B^{(0, \dots, 1, \dots, 0)}$$

(we place 1 in i -th position)

$$C_{j + \frac{(2n-i)(i-1)}{2} + n} = C^{(0, \dots, 1, \dots, 1, \dots, 0)}$$

(we place 1 in i -th and j -th position, $j < i$)

$$C_{i + \frac{(2n-i)(i-1)}{2} + n} = C^{(0, \dots, 2, \dots, n)}$$

(we place 2 in i -th position).

§3. Imbedding of T_n FM into $FF^2 M$.

Let $FM(M, \pi_M, Gl(n))$ be the frame bundle of M , $T_n^2 FM(T_n^2 M, T_n^2 \pi_M, T_n^2 Gl(n))$ the induced bundle and $FT_n^2 M(T_n^2 M, \pi_{T_n^2 M}, Gl(N))$ the frame bundle of $T_n^2 M$.

Theorem 3.1. *There exists a canonical injective homomorphism of principal bundles*

$$j_M^2: T_n^2 FM \rightarrow FT_n^2 M$$

over the identity of $T_n^2 M$, with associate Lie group homomorphism $\varrho_n^2: T_n^2 Gl(n) \rightarrow Gl(N)$.

PROOF. Let U be a coordinate neighbourhood in M , and

$$\varphi_U: T_n^2 U \times T_n^2 Gl(n) \rightarrow T_n^2 FU, \quad \psi_U: T_n^2 U \times Gl(N) \rightarrow FT_n^2 U$$

the local trivializations of $T_n^2 FM$ and $FT_n^2 M$, respectively. Then we define $j_{M|U}^2: T_n^2 FU \rightarrow FT_n^2 U$ as the composition $j_{M|U}^2 = \psi_U \circ (1_{T_n^2 U} \times \varrho_n^2) \circ \varphi_U^{-1}$. In order to prove Theorem 3.1 it is sufficient to check the following identity

$$(3.1) \quad \tilde{J}_{UV} = \varrho_n^2 \circ T_n^2 J_{UV} \quad \text{on} \quad T_n^2 U \cap T_n^2 \bar{U},$$

where $J_{UV}: U \cap \bar{U} \rightarrow Gl(n)$ and $\tilde{J}_{UV}: T_n^2 U \cap T_n^2 \bar{U} \rightarrow Gl(N)$ denote the Jacobian matrices of change of coordinates in M and $T_n^2 M$, respectively.

Let (x^i) and (y^i) be the coordinate functions in U and $\bar{U}$, respectively. We assume that $y^i = f^i(x^j)$ in $U \cap \bar{U}$. Let $j_0^2 g \in T_n^2 U \cap T_n^2 \bar{U} = T_n^2(U \cap \bar{U})$ and $(x^i, x_{(\lambda)}^i, x_{(\nu)}^i)$ be the induced coordinates. Then $T_n^2 J(j_0^2 g) = j_0^2(J \circ g)$ is the 2-jet corresponding to the composition $(J \circ g)(t) = \left(\frac{\partial f^i}{\partial x^j} \right) |_{g(t)}$. Therefore, $j_0^2(J \circ g)$ is identified to the $\left(1 + n + \frac{n(n+1)}{2}\right)$ -tupla

$$(a; \dots, B^{(\lambda)}, \dots; \dots, C^{(\nu)}, \dots),$$

where

$$(3.2) \quad \begin{aligned} a &= \left(\frac{\partial f^i}{\partial x^j} \Big|_x \right), \quad x = g(0), \\ B^{(\lambda)} &= \left(\frac{\partial^2 f^i}{\partial x^j \partial x^r} \Big|_x x_{(\lambda)}^r \right), \quad |\lambda| = 1, \\ C^{(v)} &= \left(\frac{1}{2} \frac{\partial^2 f^i}{\partial x^j \partial x^r \partial x^s} \Big|_x \sum_{\substack{\lambda \in N_2 \\ |\lambda|=1}} x_{(v-\lambda)}^r x_{(\lambda)}^s + \frac{\partial^2 f^i}{\partial x^j \partial x^r} x_{(v)}^r \right), \quad |v| = 2. \end{aligned}$$

Now, from (3.2) one gets that the matrix $Q_n^2(j_0^2(J \circ g))$ coincides with the Jacobian matrix (1.2). This proves the equality (3.1). $\#$

Now, let $T_n^2 F^2 M|_{F^2 M}$ be the restriction of $T_n^2 F^2 M$ to the open submanifold $F^2 M$ of $T_n^2 M$. Remark that the restriction $FT_n^2 M|_{F^2 M}$ is canonically isomorphic to the frame bundle $FF^2 M$ of $F^2 M$. Then, from Theorem 3.1, we deduce

Theorem 3.2. j_M^2 induces a bundle homomorphism of $T_n^2 FM|_{F^2 M}$ into $FF^2 M$ over the identity of $F^2 M$ and with associate Lie group homomorphism Q_n^2 . $\#$

§ 4. Prolongations of G -structures to $F^2 M$.

Let G be a subgroup of $Gl(n)$, and denote $G^{(2)} = Q_n^2(T_n^2 G)$. Then $G^{(2)}$ is a Lie subgroup of $Gl(N)$ isomorphic to $T_n^2 G$. Let $P(M, \pi, G)$ be a G -structure on M . Then we have

Theorem 4.1. If M has a G -structure P , then $F^2 M$ has a canonical $G^{(2)}$ -structure $P^{(2)}$.

PROOF. Taking into account Theorem 3.2, it suffices to set $P^{(2)} = j_M^2(T_n^2 P|_{F^2 M})$. $\#$

Definition 4.2. $P^{(2)}$ will be called the prolongation of the G -structure P on M to the frame bundle of second order $F^2 M$.

Let M and M' be n -dimensional manifolds, $f: M \rightarrow M'$ a diffeomorphism and $f^1: FM \rightarrow FM'$ the induced isomorphism of principal bundles (see [1]). Then we have

Theorem 4.3. The following diagram is commutative

Theorem 4.3. The following diagram is commutative

$$\begin{array}{ccc} T_n^2 FM & \xrightarrow{j_M^2} & FT_n^2 M \\ \downarrow T_n^2(f^1) & & \downarrow (T_n^2 f)^1 \\ FM & \xrightarrow{f^1} & FM' \end{array} \quad \#$$

Theorem 4.4. Let P and P' be G -structures on M and M' , respectively, and $f: M \rightarrow M'$ a diffeomorphism. Then f is an isomorphism of P to P' if and only if f^2 is an isomorphism of $P^{(2)}$ to $P'^{(2)}$.

PROOF. Suppose that f is an isomorphism of P to P' . Then by virtue of Theorem 4.3 and since $T_n^2(f^1)(T_n^2 P|_{F^2 M}) = T_n^2 P'|_{F^2 M}$, we have

$$(f^2)^1(P^{(2)}) = (f^2)^1(j_M^2(T_n^2 P|_{F^2 M})) = j_{M'}^2(T_n^2(f^1)(T_n^2 P|_{F^2 M})) = j_{M'}^2(T_n^2 P'|_{F^2 M}) = P'^{(2)}.$$

Conversely, if

$$(f^2)^1(P^{(2)}) = P'^{(2)}, \quad \text{then} \quad (f^2)^1(j_M^2(T_n^2 P|_{F^2 M})) = j_{M'}^2(T_n^2 P'|_{F^2 M}).$$

On the other hand,

$$(f^2)^1(j_M^2(T_n^2 P|_{F^2 M})) = j_{M'}^2(T_n^2(f^1)(T_n^2 P|_{F^2 M})).$$

Then, since j_M^2 is injective, we deduce that $T_n^2(f^1)(T_n^2 P|_{F^2 M}) = T_n^2 P'|_{F^2 M}$. Hence $f^1(P) = P'$ that is, f is an isomorphism of P to P' . $\#$

Corollary 4.5. *Let P be a G -structure on M and let f be a diffeomorphism of M into itself. Then f is an automorphism of P if and only if f^2 is an automorphism of $P^{(2)}$. $\#$*

Corollary 4.6. *A vector field X on M is an infinitesimal automorphism of a G -structure P on M if and only if X^C is an infinitesimal automorphism of the prolongation $P^{(2)}$ of P to $F^2 M$.*

PROOF. It is a consequence of the following result (see Gancarzewicz [3]): If φ_t is the local 1-parameter group generated by X then φ_t^2 is the local 1-parameter group generated by X^C . $\#$

§ 5. Integrability of the prolongation of G -structures

In this section, we shall prove that the prolongation of an integrable G -structure is also integrable, and viceversa.

Definition 5.1. Let $P(M, \pi, G)$ be a G -structure on M . P is said to be integrable if for each point $x \in M$ there is a coordinate system (U, x^i) with $x \in U$ such that the frame $\left(\frac{\partial}{\partial x^1}\Big|_y, \dots, \frac{\partial}{\partial x^n}\Big|_y\right) \in P$ for every $y \in U$.

Theorem 5.2. Let (U, x^i) be a coordinate system in M , and let $\Phi: U \rightarrow FM$ be a cross-section given by $\Phi(x) = \left(\Phi_j^i(x) \frac{\partial}{\partial x^i}\Big|_x\right)$, $x \in U$. Define $\Phi^2 = j_M^2 \circ T_n^2 \Phi: T_n^2 U \rightarrow FT_n^2 M$. Then Φ^2 is also a cross-section which is given at $p \in T_n^2 U$, with $\pi_n^2(p) = x$, by

$$(5.1) \quad \Phi^2(p) = (X_{i|p}^C, X_{i|p}^{(\lambda)}, X_{i|p}^{(v)}), \quad i = 1, \dots, n; \lambda, v \in N_2, |\lambda| = 1, |v| = 2,$$

where X_i is the local vector field given on U by $X_{i|x} = \Phi_i^j(x) \frac{\partial}{\partial x^j}\Big|_x$, $x \in U$, and X_i^C , $X_i^{(\lambda)}$, $X_i^{(v)}$ are the complete lift, (λ) -lift and (v) -lift of X_i to $T_n^2 M$.

PROOF. From Theorem 3.1, one easily prove that Φ^2 is a cross-section. Now, putting $f(x) = (\Phi_j^i(x)) \in GL(n)$, for $x \in U$, then we have $\varphi_U^{-1} \circ \Phi = (1_U, f)$, where φ_U is the local trivialization of FM . Hence

$$\Phi^2 = \psi \circ (1_{T_n^2 M} \times \varrho_n^2) \circ T_n^2(1_U, f) = \psi_U \circ (1_{T_n^2 M} \times (\varrho_n^2 \circ T_n^2 f)),$$

where φ_U is the local trivialization of $T_n^2 FM$.

If $j_0^2 h \in T_n^2 U$ has coordinates $(x^i, x_{(\lambda)}^i, x_{(v)}^i)$ then $T_n^2 f(j_0^2 h)$ has coordinates

$$\left(\Phi_j^i(x), \frac{\partial \Phi_j^i}{\partial x^r} \Big|_x x_{(\lambda)}^r, \frac{1}{2} \frac{\partial^2 \Phi_j^i}{\partial x^r \partial x^s} \Big|_x \sum_{\substack{\lambda \in N_2 \\ |\lambda|=1}} x_{(v-\lambda)}^r x_{(\lambda)}^s + \frac{\partial \Phi_j^i}{\partial x^r} \Big|_x x_{(v)}^r \right)$$

Then, from Proposition 2.1, one deduces (5.1). $\#$

Actually, if $\Phi: U \rightarrow FM$ is a cross-section, then the restriction $\bar{\Phi} = \Phi^2|_{F^2 U}: F^2 U \rightarrow FF^2 M$ is also a cross-section locally expressed by

$$(5.2) \quad \bar{\Phi}(p) = (X_{i|p}^c, X_{i|p}^{(\lambda)}, X_{i|p}^{(v)}), \quad i = 1, \dots, n; \quad \lambda, v \in N_2, \quad |\lambda| = 1, |v| = 2,$$

where $X_i = \Phi_i^j \frac{\partial}{\partial x^j}$ in U .

Proposition 5.3. *Let P be a G -structure on M . Then P is integrable if and only if the prolongation $P^{(2)}$ is integrable.*

PROOF. Suppose that P is integrable. From Definition 5.1 and Theorem 5.2 we deduce that $P^{(2)}$ is integrable.

Conversely, suppose that $P^{(2)}$ is integrable. Let x_0 be an arbitrary point in M , (U, x^i) a coordinate system with $x_0 \in U$, and $\Phi: U \rightarrow P$ a local cross-section of P over U . Now, let $p_0 \in F^2 U$ be with coordinates $(x^i = x^i(x_0), x_{(\lambda)}^i = \delta^{i\lambda}, x_{(v)}^i = 0)$. Since $P^{(2)}$ is integrable there exists a coordinate system $(\tilde{U}, y^i, y_{(\lambda)}^i, y_{(v)}^i): i = 1, \dots, n; |\lambda| = 1, |v| = 2$ in $F^2 U$ with $p_0 \in \tilde{U}$, $\tilde{U} \subset F^2 U$, such that, if we define $\tilde{\Phi}_0$ by

$$\tilde{\Phi}_0(p) = \left(\frac{\partial}{\partial y^i} \Big|_p, \frac{\partial}{\partial y_{(\lambda)}^i} \Big|_p, \frac{\partial}{\partial y_{(v)}^i} \Big|_p \right),$$

then $\tilde{\Phi}_0$ is a cross-section of $P^{(2)}$ over $\tilde{U}$. Now, since $\bar{\Phi}$ and $\tilde{\Phi}_0$ are both cross-sections of $P^{(2)}$ over $\tilde{U}$, there exists a map $\tilde{g}: \tilde{U} \rightarrow G^{(2)} = \varrho_n^2(T_n^2 G)$ such that $\tilde{\Phi}_0(p) = \bar{\Phi}(p) \cdot \tilde{g}(p)$ holds for $p \in \tilde{U}$. Then, using similar arguments as in the proof of Proposition 5.5 [7], Proposition 10. [8] and Proposition 4.5 [1], we deduce that P is integrable. $\#$

§ 6. Prolongations of classical G -structures

Let P be a G -structure on M , (U, x^i) a local coordinate system in U , and $\Phi: U \rightarrow P$ a cross-section. Then Φ defines a local field of frames $\{X_1, \dots, X_n\}$ adapted to P and given by $X_i = \Phi_i^j \frac{\partial}{\partial x^j}$. Hence the local field of coframes $\{\theta^1, \dots, \theta^n\}$ dual to $\{X_1, \dots, X_n\}$ is given by

$$(6.1) \quad \theta^j = \psi_i^j dx^i,$$

where (ψ_i^j) denotes the inverse matrix of (Φ_j^i) . Then Φ induces a cross-section $\bar{\Phi}: U \rightarrow P^{(2)}$ given by (5.2). There $\bar{\Phi}$ defines the local field of frames adapted to $P^{(2)}$ given by $\{X_i^c, X_i^{(\lambda)}, X_i^{(v)}: i=1, \dots, n; \lambda, v \in N_2, |\lambda|=1, |v|=2\}$. From (6.1), we deduce that the dual local field of coframes is $\{(\theta^i)^v, (\theta^i)^{(\lambda)}, (\theta^i)^{(\nu)}: i=1, \dots, n; \lambda, v \in N_2, |\lambda|=1, |v|=2\}$.

(I) *G*-structures defined by tensor fields of type (1, 1)

Let $\varrho: Gl(n) \rightarrow \text{Aut}(R^n)$ be the canonical representation of $Gl(n)$ into R^n , $u \in \text{End}(R^n)$ an arbitrary element and G_u the isotropy group of u with respects to ϱ . Let $u^2 = T_n^2 u \in \text{End}(R^N)$. R^N identified to $T_n^2 R^n$, the induced map defined by $u^2(j_0^2 g) = j_0^2(u \circ g)$, $j_0^2 g \in T_n^2 R^n$. Let $(x^i, x_{(\lambda)}^i, x_{(v)}^i)$ and $(x^i, x_{(\lambda)}^i, x_{(v)}^i)$ the induced coordinates of $j_0^2(u \circ g)$ and $j_0^2 g$, respectively. If $u = (u_j^i)$ is the matrix representation of u , then we obtain

$$x^i = u_j^i x^j, x_{(\lambda)}^i = u_j^i x_{(\lambda)}^j, x_{(v)}^i = u_j^i x_{(v)}^j, \lambda, v \in N_2, |\lambda|=1, |v|=2,$$

and therefore the matrix representation of u^2 is

$$(6.2) \quad u^2 = \begin{pmatrix} (u_j^i) & 0 \\ & \ddots \\ 0 & (u_j^i) \end{pmatrix}$$

From (6.2) we deduce

Lemma 6.1. *Let $u^2 = T_n^2 u \in \text{End}(R^N)$ be the linear map induced by $u \in \text{End}(R^n)$. If $\text{rank } u = r$, then $\text{rank } u^2 = r \left(1 + n + \frac{n(n+1)}{2}\right)$. Moreover, if u satisfies a polynomial equation $Q(u) = 0$, then u^2 satisfies the same equation, that is, $Q(u^2) = 0$. #*

Proposition 6.2. *Let G_{u^2} be the isotropy group of u^2 with respect to the canonical representation of $Gl(N)$ into R^N , and denote $(G_u)^{(2)} = \varrho_n^2(T_n^2 G_u)$. Then $(G_u)^{(2)} \subset G_{u^2}$. #*

Theorem 6.3. *If M admits a G_u -structure, then $F^2 M$ admits a G_{u^2} -structure. Moreover, if the G -structure in M is integrable, then the induced G_{u^2} -structure on $F^2 M$ is so also.*

PROOF. From Theorem 4.1 we deduce that $F^2 M$ admits a $(G_u)^{(2)}$ -structure, which, by virtue of Proposition 6.2, can be extended to a G_{u^2} -structure. The assertion on the integrability follows from Proposition 5.3. #

Let P be a G_u -structure on M , and let F be the tensor field of type (1, 1) on M associated to P . If (U, x^i) is a coordinate system in M , and if $\{X_i\}, \{\theta^i\}$ are the local field of frames and coframes induced by a cross-section $\Phi: U \rightarrow P$, then F is locally given by

$$(6.3) \quad F = F_j^i \frac{\partial}{\partial x^i} \otimes dx^j = u_j^i X_i \otimes \theta^j.$$

Similarly, let $F^{(2)}$ be the tensor field of type (1, 1) on $F^2 M$ associated to $\bar{P}^{(2)}$,

extension of the prolongation $P^{(2)}$ of P (Theorem 6.3). Then, from (5.2) and (6.2), we have

$$(6.4) \quad F^{(2)} = u_j^i X_i^C \otimes (\theta^j)^V + \sum_{|\lambda|=|\mu|=1} \delta^{\lambda\mu} u_j^i X_i^{(\lambda)} \otimes (\theta^j)^{(\mu)} + \sum_{|\nu|=|\eta|=2} \delta^{\nu\eta} u_j^i X_i^{(\nu)} \otimes (\theta^j)^{(\eta)}.$$

From (6.4), one easily deduces that $F^{(2)} = F^C$.

Summing up, we can state

Theorem 6.4. *Let be $u \in \text{End}(R^n)$, P a G_u -structure on M , and F the tensor field of type $(1, 1)$ induced by P on M . Then the complete lift F^C of F to $F^2 M$ defines the G_{u^2} -structure on $F^2 M$ given in Theorem 6.3. $\#$*

From Lemma 6.1 and Theorem 6.4, we deduce

Corollary 6.5. [3]. *If F defines on M a polynomial structure of rank r , then F^C defines on $F^2 M$ a polynomial structure of rank $r \left(1 + n + \frac{n(n+1)}{2}\right)$ and same structural polynomial. $\#$*

(II) G -structures defined by tensor fields of type $(0, 2)$

Let be $u \in \otimes_2(R^n)^*$, $T_n^2 u: T_n^2 R^n \times T_n^2 R^n \rightarrow T_n^2 R$, the induced map, and let $p_1, p_2: T_n^2 R \cong R^{N/n} \rightarrow R$ be the maps defined by

$$p_1(j_0^2 g) = \sum_{\substack{\lambda \in N_n^2 \\ |\lambda|=1}} C_{(\lambda)}, \quad p_2(j_0^2 g) = \sum_{\substack{\nu \in N_n^2 \\ |\nu|=2}} D_{(\nu)}$$

respectively, where $(g(0), C_{(\lambda)}, D_{(\nu)})$ are the induced coordinates of $j_0^2 g \in T_n^2 R$, $g: R^n \rightarrow R$. Then we define

$$\tilde{u}_i^2: R^N \times R^N \rightarrow R$$

as the composition $\tilde{u}_i^2 = p_i \circ T_n^2 u$, $i=1, 2$.

Lemma 6.6. $\tilde{u}_i^2 \in \otimes_2(R^N)^*$, that is, $\tilde{u}_i^2$ is bilinear, $i=1, 2$. Moreover, if u is symmetric (resp. skew-symmetric) then $\tilde{u}_i^2$ is also symmetric (resp. skew-symmetric) and if rank $u=r$, then rank $\tilde{u}_1^2=2r$ and rank $\tilde{u}_2^2=3r$.

PROOF. Suppose that (u_{ij}) is the matrix representation of u . Let be $j_0^2 f, j_0^2 g \in T_n^2 R^n \cong R^N$ with the induced coordinates $(x^i, x_{(\lambda)}^i, x_{(\nu)}^i)$ and $(y^i, y_{(\lambda)}^i, y_{(\nu)}^i)$, respectively. Then, since $u_1^2(j_0^2 f, j_0^2 g) = p_1(j_0^2(u \circ (f, g)))$ and $u \circ (f, g)(t) = f^i(t) u_{ij} g^j(t)$, $t \in R^n$, we have

$$\begin{aligned} \tilde{u}_1^2(j_0^2 f, j_0^2 g) &= \sum_{\substack{\lambda \in N_n^2 \\ |\lambda|=1}} (x_{(\lambda)}^i u_{ij} y^j + x^i u_{ij} y_{(\lambda)}^j) \\ \tilde{u}_2^2(j_0^2 f, j_0^2 g) &= \sum_{\substack{\nu \in N_n^2 \\ |\nu|=1}} \{x_{(\nu)}^i u_{ij} y^j + \sum_{\substack{\lambda \in N_n^2 \\ |\lambda|=1}} x_{(\nu-\lambda)}^i u_{ij} y_{(\lambda)}^j + x^i u_{ij} y_{(\nu)}^j\} \end{aligned}$$

Therefore, the matrix representations of $\tilde{u}_1^2$ and $\tilde{u}_2^2$ are

(6.5)

$$\tilde{u}_1^2 = \begin{pmatrix} 0 & (u_{ij}) \dots (u_{ij}) & 0 \dots 0 \\ (u_{ij}) & & \\ \vdots & 0 & 0 \\ (u_{ij}) & & \\ 0 & & \\ \vdots & 0 & 0 \\ 0 & & \end{pmatrix}, \quad \tilde{u}_2^2 = \begin{pmatrix} 0 & 0 \dots 0 & (u_{ij}) \dots (u_{ij}) \\ 0 & (u_{ij}) \dots (u_{ij}) & \\ \vdots & \vdots & \vdots & 0 \\ 0 & (u_{ij}) \dots (u_{ij}) & \\ (u_{ij}) & & \\ \vdots & 0 & 0 \\ (u_{ij}) & & \end{pmatrix},$$

respectively. Lemma 6.6 is now obvious. $\#$

Proposition 6.7. *Let G_u (resp. $G_{\tilde{u}_i^2}$) be the isotropy group of $u \in \otimes_2(R^n)^*$ (resp. $\tilde{u}_i^2 \in \otimes_2(R^n)^*$) with respect to the canonical representation of $Gl(n)$ into R^n (resp. of $Gl(N)$ into R^N), and denote $(G_u)^{(2)} = \varrho_n^2(T_n^2 G_u)$. Then*

$$(G_u)^2 \subset G_{\tilde{u}_i^2}, \quad i = 1, 2.$$

PROOF. Direct from (6.5). $\#$

Theorem 6.8. *If M admits a G_u -structure, then $F^2 M$ admits a $G_{\tilde{u}_i^2}$ -structure, $i=1, 2$. Moreover, if the G_u -structure is integrable then the induced $G_{\tilde{u}_i^2}$ -structure is so also, $i=1, 2$. $\#$*

Corollary 6.9. *If M has an almost symplectic (resp. symplectic) structure then $F^2 M$ has two induced almost presymplectic (resp. presymplectic) structures. $\#$*

Let be $u \in \otimes_2(R^n)^*$, P a G_u -structure on M , and $\tilde{P}_i^{(2)}$, $i=1, 2$, the induced $G_{\tilde{u}_i^2}$ -structure on $F^2 M$, and g (resp. $\tilde{g}_i^{(2)}$, $i=1, 2$) the tensor field of type $(0, 2)$ on M (resp. on $F^2 M$) associated to P (resp. to $\tilde{P}_i^{(2)}$, $i=1, 2$). Then we have in U

$$g = g_{ij} dx^i \otimes dx^j = u_{ij} \theta^i \otimes \theta^j$$

and from (6.5) and (6.6), we have in $F^2 U$

$$\tilde{g}_1^{(2)} = \sum_{\substack{\lambda \in N_{\frac{n}{2}} \\ |\lambda|=1}} \{u_{ij}(\theta^i) \otimes (\theta^j)^{(\lambda)} + u_{ij}(\theta^i) \otimes (\theta^j)^{(\nu)}\},$$

$$\tilde{g}_2^{(2)} = \sum_{\substack{\nu \in N_{\frac{n}{2}} \\ |\nu|=2}} \{u_{ij}(\theta^i)^{\nu} \otimes (\theta^j)^{(\nu)} + \sum_{\substack{\lambda \in N_{\frac{n}{2}} \\ |\lambda|=1}} u_{ij}(\theta^i)^{(\nu-\lambda)} \otimes (\theta^j)^{(\lambda)} + u_{ij}(\theta^i)^{(\nu)} \otimes (\theta^j)^{(\nu)}\}$$

Then one easily deduces

$$(6.7) \quad \tilde{g}_1^{(2)} = \sum_{\substack{\lambda \in N_{\frac{n}{2}} \\ |\lambda|=1}} g^{(\lambda)}, \quad \tilde{g}_2^{(2)} = \sum_{\substack{\nu \in N_{\frac{n}{2}} \\ |\nu|=2}} g^{(\nu)}.$$

Remark. Let $\pi_1^2: F^2M \rightarrow FM$ be the projection defined by $\pi_1^2(j_0^2 f) = j_0^1 f$. Then, from (6.7) one obtains that $\tilde{g}_1^{(2)} = (\pi_1^2)^* g^C$, where g^C is the complete lift of g to FM defined by Mok [5].

As in [1], [7] and [8] one can obtain prolongations of $Gl(V, W)$ and $Sl(n, R)$ -structures on M to F^2M .

§ 7. Lifts of G -connections to F^2M

Let G be a Lie subgroup of $Gl(n)$, $P(M, \pi, G)$ a G -structure on M and ∇ a linear connection on M . Let U be an arbitrary coordinate neighbourhood on M and $\{X_i\}$ a local field of frames adapted to P . Then, for any vector field Y on M assume that

$$(7.1) \quad \nabla_Y X_i = Y^k A_{ki}^h X_h$$

holds, the matrix $(Y^k A_{ki}^h)$ belonging to the Lie algebra $\underline{G}$ of G , where $Y = Y^k X_k$. Under these assumptions, ∇ is said to be a G -connection relative to the G -structure P and the coefficients A_{ki}^h in (7.1) are called the components of ∇ with respect to the adapted frame $\{X_i\}$.

Now, we consider the so called complete lift ∇^C to $T_n^2 M$ of a linear connection ∇ on M , which is defined as the unique linear connection on $T_n^2 M$ verifying

$$\nabla_{X^{(\alpha)}}^C Y^{(\beta)} = (\nabla_X Y)^{(\alpha+\beta)},$$

for any vector fields X and Y on M , $\alpha, \beta \in N_2$ (see MORIMOTO [9]). Now, we consider the restriction of ∇^C to F^2M , also denoted by ∇^C . Then ∇^C is characterized by the following identities

$$(7.2) \quad \begin{aligned} \nabla_{X^C}^C Y^C &= (\nabla_X Y)^C, \quad \nabla_{X^C}^C Y^{(\lambda)} = \nabla_{X^{(\lambda)}}^C Y^C = (\nabla_X Y)^{(\lambda)}, \\ \nabla_{X^C}^C Y^{(\nu)} &= \nabla_{X^{(\nu)}}^C Y^C = (\nabla_X Y)^{(\nu)}, \quad \nabla_{X^{(\lambda)}}^C Y^{(\mu)} = (\nabla_X Y)^{(\lambda+\mu)}, \\ \nabla_{X^{(\lambda)}}^C Y^{(\nu)} &= \nabla_{X^{(\nu)}}^C Y^{(\lambda)} = \nabla_{X^{(\eta)}}^C Y^{(\eta)} = 0, \end{aligned}$$

for any vector fields X and Y on M , and every $\lambda, \mu, \nu, \eta \in N_2$ such that $|\lambda| = |\mu| = 1$, $|\nu| = |\eta| = 2$.

Definition 7.1. ∇^C is called the complete lift of ∇ to F^2M .

Now, if X_i is a local field of frames on U adapted to a G -structure P , then $\{X_i^C, X_i^{(\lambda)}, X_i^{(\nu)}\}$ is a local field of frames on F^2U adapted to the prolongation $P^{(2)}$ of P to F^2M . Hence, from (7.1) and (7.2), we obtain

$$\begin{aligned} \nabla_{X_i^C}^C X_j^C &= (A_{ij}^h)^V X_h^C + \sum_{\substack{\lambda \in N_2 \\ |\lambda|=1}} (A_{ij}^h)^{(\lambda)} X_h^{(\lambda)} + \sum_{\substack{\nu \in N_2 \\ |\nu|=2}} (A_{ij}^h)^{(\nu)} X_h^{(\nu)}, \\ \nabla_{X_i^C}^C X_j^{(\lambda)} &= \nabla_{X_i^{(\lambda)}}^C X_j^C = (A_{ij}^h)^V X_h^{(\lambda)} + \sum_{\substack{\mu \in N_2 \\ |\mu|=2}} (A_{ij}^h)^{(\mu)} X_h^{(\mu+\lambda)}, \\ \nabla_{X_i^C}^C X_j^{(\nu)} &= \nabla_{X_i^{(\nu)}}^C X_j^C = (A_{ij}^h)^V X_h^{(\nu)}, \\ \nabla_{X_i^{(\lambda)}}^C X_j^{(\nu)} &= \nabla_{X_i^{(\nu)}}^C X_j^{(\lambda)} = \nabla_{X_i^{(\eta)}}^C X_j^{(\eta)} = 0. \end{aligned}$$

Therefore, we deduce

Theorem 7.2. *Let ∇ be a G -connection relative to a G -structure P on M . Then the complete lift ∇^C of ∇ to F^2M is a $G^{(2)}$ -connection relative to the prolongation $P^{(2)}$ of P to F^2M . #*

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