

Morita equivalence for a larger class of rings

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In [10] it was observed that two rings S and R with unities are Morita equivalent if and only if there exists a gamma ring with right and left unities such that its right and left operator rings are isomorphic to S and R . This has been extended to rings with local units [11]. In this paper we extend this result to rings satisfying the conditions i) $S^2=S$ and ii) $aS=0$ or $Sa=0$ implies that $a=0$, $a \in S$ which include rings with unities and rings with local units.*) We also prove that any ring satisfying the above conditions is equivalent to a division ring if and only if it is simple and completely reducible, thus obtaining an extension of Wedderburn theorem for simple Artinian rings.

1. Preliminaries

Let A and Γ be additive abelian groups. Then following BARNES [3] we say A is a Γ -ring if there exists a map $f: A \times \Gamma \times A \rightarrow A$ with $f(x, \alpha, y) = x\alpha y$ such that

i) f is additive in each variable, and

ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in A$ and $\alpha, \beta \in \Gamma$.

Consider the maps $[\alpha, x]: y \rightarrow y\alpha x$ and $[x, \alpha]: y \rightarrow x\alpha y$ $x \in A, \alpha \in \Gamma$ and for all $y \in A$. Clearly $[x, \alpha], [\alpha, x]$ belong to the endomorphism group $\text{End}(A)$. The bilinearity of the map $\Gamma \times A \rightarrow \text{End } A$ ($A \times \Gamma \rightarrow \text{End } A$) given by $(\alpha, a) \rightarrow [\alpha, a]$ ($(a, \alpha) \rightarrow [a, \alpha]$) gives rise to a linear map from $\Gamma \otimes_Z A \rightarrow \text{End } A$ ($A \otimes_Z \Gamma \rightarrow \text{End } A$) given by

$$\sum_i \alpha_i \otimes a_i \rightarrow \sum_i [\alpha_i, a_i] \quad (\sum_i a_i \otimes \alpha_i \rightarrow \sum_i [a_i, \alpha_i]),$$

$\alpha_i \in \Gamma$ and $a_i \in A$. The image of $\Gamma \otimes_Z A$ ($A \otimes_Z \Gamma$) in $\text{End } A$ is an associative ring denoted by $R(A, \Gamma)$ ($L(A, \Gamma)$) and call it the right (left) operator ring of A . Ring multiplication in $R(A, \Gamma)$ and $L(A, \Gamma)$ is given by the rule,

$$\sum_i [\alpha_i, a_i] \sum_j [\beta_j, b_j] = \sum_{i,j} [\alpha_i, \alpha_i \beta_j b_j],$$

*) See about T. W. ANDERSON—K. R. FULLER, Rings and Categories of Modules, Springer (1973)

and

$$\sum_i [a_i, \alpha_i] \sum_j [b_j, \beta_j] = \sum_{i,j} [a_i \alpha_i b_j, \beta_j].$$

A is clearly a faithful $R(A, \Gamma) - L(A, \Gamma)$ bimodule.

For further details on Γ -rings and for literature on ring theory we refer to [3], [6], [7], [10] and [2], [5] respectively.

2. Throughout this paper we denote a Γ -ring by (A, Γ) or A , right and left operator rings by $R(A, \Gamma)$ or R and $L(A, \Gamma)$ or L respectively. All rings considered satisfy the gamma conditions namely,

- (i) $S^2 = S$,
- (ii) $Sa = 0$ or $aS = 0$ implies $a = 0$, $a \in S$.

Definition 2.1 [12]. A Γ -ring A is said to be weakly semiprime if $[x, \Gamma] \neq 0$ and $[\Gamma, x] \neq 0$ for all $0 \neq x \in A$ and $A\Gamma A = A$.

Remark. A is weakly semiprime implies $R(A, \Gamma)$, $L(A, \Gamma)$ satisfy the gamma conditions. Any ring with unity satisfies the gamma conditions, Fuller's rings with "enough idempotents" and more generally a ring with local units [1] satisfy the gamma conditions.

The following is a ring which satisfies the gamma conditions but does not contain unity or idempotent and hence has no local units.

Let $B \subset \mathcal{C}[0, 1]$ be the set of all continuous real valued functions on the interval $[0, 1]$ which vanish in a neighbourhood of 0. B is a commutative ring under the pointwise addition and pointwise multiplication. The ring B can be easily checked to satisfy the gamma conditions.

Now we state the results which are needed for our purpose.

Lemma 2.2. [12]. Let A be a weakly semiprime Γ -ring, L and R its operator rings. Then A is simple if and only if $R(L)$ is simple.

Theorem 2.3. (Theorem 1 of [9].) Let A be a weakly semiprime Γ -ring, L and R be its operator rings. Then L and R are Morita Equivalent.

We now prove the converse. But first we need the following lemma

Lemma 2.4. Let R and S be two associative rings satisfying the gamma conditions and R be Morita equivalent to S . Suppose that

$$H: \mathcal{C}_R^* \rightarrow \mathcal{C}_S \quad \text{and} \quad T: \mathcal{C}_S \rightarrow \mathcal{C}_R$$

are category equivalences with $HT \cong 1_{\mathcal{C}_S}$ and $TH \cong 1_{\mathcal{C}_R}$. Then ${}_S T(S)_R$ and ${}_R H(R)_S$ are canonical bimodules and

- (1) H and T are full and faithful;
- (2) (H, T) and (T, H) are adjoint pairs of functors;
- (3) $H \cong (\text{Hom}_R(T(S), -))_S$, $T \cong (\text{Hom}_S(H(R), -))_R$;
- (4) $S \cong (\text{End}_R(T(S)))_S$;
- (5) $T(S)$ is an R -generator in $\mathcal{C}_R$ and $H(R)$ is an S -generator in $\mathcal{C}_S$.

*) $\mathcal{C}_R$ denotes the subcategory of right R -modules with spanning conditions, $MR = M$ and $mR = 0$ implies $m = 0$, $m \in M$.

PROOF. 1) and (2) can be proved as in [2], (3) Let $M \in \mathcal{C}_R$. We define a map

$$\eta: M \rightarrow (\text{Hom}_R(R, M))R \text{ by}$$

$$m \rightarrow \eta(m) \text{ such that } (\eta(m))(r) = mr.$$

It is straightforward to prove that η is an R -module isomorphism.

Also $H(M) \in \mathcal{C}_S$. Hence using the above isomorphism we have

$$H(M) \cong (\text{Hom}_S(S, H(M)))S \cong \text{Hom}_S(T(S), M)S$$

So $H \cong (\text{Hom}_R(T(S), -))S$. Similarly we prove $T \cong (\text{Hom}_S(H(R), -))R$.

(4) By (iii) of above, we have,

$$H(T(S)) \cong (\text{Hom}_R(T(S), T(S)))S = \text{End}_R(T(S))S.$$

(5) Let $M \in \mathcal{C}_R$. We define a map

$$\omega: R^{(M)} \rightarrow M \text{ by}$$

$$r_{1m_1} + \dots + r_{tm_t} \rightarrow m_1r_1 + \dots + m_tm_t.$$

Since $MR = M$, ω is an epimorphism and R generates $\mathcal{C}_R$.

Next let $N \in \mathcal{C}_S$, then there is an epimorphism $R^{(X)} \rightarrow T(N) \rightarrow 0$ in $\mathcal{C}_R$ where X is an indexing set. Since H is a category equivalence, it preserves epimorphisms and direct sums and hence we have an epimorphism $(H(R))^{(X)} \rightarrow H(T(N)) \rightarrow 0$ in $\mathcal{C}_S$. So $H(R)$ generates $\mathcal{C}_S$.

Similarly it can be shown that $T(S)$ is an R -generator.

This completes the proof of the lemma.

From (3) of Lemma 2.4, we have $T(S) \cong (\text{Hom}_S(H(R), S))R$ as right R -modules and $H(R) \cong \text{Hom}_R(S, H(R))S$ as right S -modules. We observe that $H(R) \cong \text{Hom}_R(S, H(R))S$ as left R -modules also. To see this, we define $(rf)(s) = rf(s)$, $r \in R, f \in \text{Hom}_R(S, H(R))S$ and if $\Theta: H(R) \xrightarrow{\cong} (\text{Hom}_S(S, H(R)))S$ denotes the isomorphism as abelian groups, given by $h \rightarrow \Theta(h)$, then $(rh) \rightarrow \Theta(rh)$ such that $(\Theta(rh))(s) = (rh)(s) = r(hs) = r(\Theta(h))(s)$. Hence $H(R) \cong (\text{Hom}_S(S, H(R)))S$ as left R -modules.

Again we have $\psi: \text{Hom}_R(T(S), R) \rightarrow \text{Hom}_S(S, H(R))$ is an abelian group isomorphism given by $\psi(f) = H(f)\eta_S$ where $f \in \text{Hom}_R(T(S), R)$ and $\eta_S: S \rightarrow HT(S)$ is an isomorphism. We show that ψ is a left R -module isomorphism. We define, for $f \in \text{Hom}_R(T(S), R)$, $g \in \text{Hom}_S(S, H(R))$, $r \in R$,

$$(rf)(x) = r(f(x)), \quad x \in T(S) \quad \text{and} \quad (rg)(s) = H(\varrho_r)g(s),$$

$s \in S$ and $\varrho_r: R \rightarrow R$ is left multiplication. Combining these we get $H(R) \overset{\varphi}{\cong} (\text{Hom}_R(T(S), R))S$ is a left R and right S bimodule isomorphism.

We are ready to prove the converse.

Theorem 2.5. *Let R and S be two rings, satisfying the gamma conditions, which are Morita equivalent. Then there exists a weakly semiprime gamma ring such that its right and left operator rings are isomorphic to R and S respectively.*

PROOF. In the notations of Lemma 2.4, we denote the canonical bimodules ${}_S T(S)_R$ and $(\text{Hom}_R(T(S), R))S$ by ${}_S A_R = {}_S T(S)_R$ and ${}_R \Gamma_S = (\text{Hom}_R(T(S), R))S$. We give a Γ -ring structure to A by defining

$$\begin{aligned} A \times \Gamma \times A &\rightarrow A \quad \text{by} \\ (a, f, a') &\rightarrow af(a'). \end{aligned}$$

Similarly we define a map from

$$\begin{aligned} \Gamma \times A \times \Gamma &\rightarrow \Gamma \quad \text{by} \\ (f, a, f') &\rightarrow f(a)f'. \end{aligned}$$

This defines on Γ , the structure of a A -ring. Clearly (A, Γ) and (Γ, A) are weakly semiprime gamma rings.

Now we prove that R is isomorphic to the right operator ring of the gamma ring (A, Γ) . But first we show that $RH(R) = R$. To see this let $\varrho_r: R \rightarrow R$ denote the left multiplication by elements of R , for every $r \in R$. Now we have epimorphisms

$$\bigoplus_{r \in R} R \xrightarrow{\oplus \varrho_r} R \rightarrow 0 \quad \text{and} \quad \bigoplus_{r \in R} H(R) \xrightarrow{\oplus H(\varrho_r)} H(R) \rightarrow 0,$$

since H is a category equivalence. So every $x \in H(R)$ can be written as $\sum_i r_i x'_i$, $r_i \in R$, $x'_i \in H(R)$. Hence $RH(R) = H(R)$. This in view of the bimodule isomorphism

$$\varphi: H(R) \rightarrow (\text{Hom}_R(T(S), R))S \quad \text{implies that} \quad R\Gamma = \Gamma.$$

Similarly it can be proved $ST(S) = T(S)$.

We define a map

$$\begin{aligned} \sigma: [\Gamma, T(S)] &\rightarrow R \quad \text{by} \\ \sigma\left(\sum_i [f_i d_i, x_i]\right) &= \sum_i (f_i d_i)(x_i) = \sum_i f_i(d_i x_i). \end{aligned}$$

Clearly σ is a well defined ring homomorphism. If $\sum_i f_i(d_i x_i) = 0$, then $\sum_i (f_i d_i)(x_i) = 0$. This implies $\sum_i [f_i d_i, x_i] = 0$. Hence σ is injective.

Next, since $T(S)$ is an R -generator in $\mathcal{C}_R$ an element $r \in R$ can be written in the form $r = \sum_i f_i(t_i)$ where $f_i \in \text{Hom}_R(T(S), R)$ and $t_i \in T(S)$. Again we have $ST(S) = T(S)$. Hence $t_i = \sum_j \delta_j y_j$, $\delta_j \in S$, $y_j \in T(S)$. So

$$r = \sum_i f_i\left(\sum_j \delta_j y_j\right) = \sum_{i,j} f_i(\delta_j y_j) = \sum_{i,j} (f_i \delta_j)(y_j).$$

Thus $r \in R$ corresponds to the element $\sum_{i,j} [f_i \delta_j, y_j]$ in $[\Gamma, T(S)]$. Hence σ is an epimorphism. That is $R \cong R(T(S), \Gamma)$.

Before we proceed to establish the isomorphism between $L(T(S), \Gamma)$ and S , we show that $S \cong (\text{End } T(S)_R)S$ as rings. We define a map

$$\Theta: S \rightarrow \text{Hom}_R(T(S), T(S))S$$

by

$$\Theta(s) = T(\eta_S^{-1})T(\varrho_{\eta_S(s)}),$$

where $s \in S$ and $\eta_S: S \rightarrow HT(S)$ is a right S -module isomorphism and $\varrho_{\eta_S(s)}$ is left multiplication.

If $s_1, s_2 \in S$, then

$$\varrho_{\eta_S(s_1)} \eta_S^{-1} \varrho_{\eta_S(s_2)} = \varrho_{\eta_S(s_1)} \varrho_{s_2}.$$

To see this, let $s' \in S$. Then

$$(\varrho_{\eta_S(s_1)} \eta_S^{-1} \varrho_{\eta_S(s_2)})(s') = \varrho_{\eta_S(s_1)}(\eta_S^{-1}(\eta_S(s_2)s')) = \eta_S(s_1)s_2s'$$

and

$$(\varrho_{\eta_S(s_1)} \varrho_{s_2})(s') = \eta_S(s_1)s_2s'.$$

So

$$T(\eta_S^{-1} \varrho_{\eta_S(s_1)} \varrho_{s_2}) = T(\eta_S^{-1} \varrho_{\eta_S(s_1)}) T(\eta_S^{-1} \varrho_{\eta_S(s_2)})$$

and hence Θ is a ring homomorphism.

This together with Lemma 2.4 (4) establishes the required ring isomorphism between S and $(\text{End } T(S)_R)S$.

The inverse isomorphism

$$\Theta^{-1}: (\text{End } T(S)_R)S \rightarrow S \text{ is given by}$$

$$\Theta^{-1}(\sum_i f_i s_i) = \sum_i \eta_S^{-1} H(f_i) \eta_S(s_i).$$

Now we define a map

$$\lambda: [T(S), \Gamma] \rightarrow S \text{ as the composition of}$$

$$[T(S), \Gamma] \xrightarrow{\mu} (\text{End } T(S)_R)S \xrightarrow{\Theta^{-1}} S$$

$$\mu(\sum_i [t_i, g_i s_i]) \rightarrow \mu(\sum_i [t_i, g_i s_i]) \rightarrow \sum_i \eta_S^{-1} H(\mu([t_i, g_i])) \eta_S(s_i)$$

where

$$(\sum_i [t_i, g_i s_i])(t') = \sum_i [t_i, g_i s_i] t', t' \in T(S).$$

$$\text{That is, } \lambda(\sum_i [t_i, g_i s_i]) = \Theta^{-1} \mu(\sum_i [t_i, g_i s_i]) = \sum_i \eta_S^{-1} H(\mu([t_i, g_i])) \eta_S(s_i).$$

$$\text{But } \mu(\sum_i [t_i, g_i])(t') = \sum_i [t_i, g_i] t' = \sum_i t_i g_i(t') = \sum_i \varrho_{t_i} g_i(t')$$

$$\text{and hence } \lambda(\sum_i [t_i, g_i s_i]) = \sum_i \eta_S^{-1} H(\varrho_{t_i}) H(g_i) \eta_S(s_i).$$

This is clearly a well defined ring monomorphism.

To see that λ is onto it suffices to check on generators.

Since $H(R)$ is an S -generator and $R\Gamma = \Gamma$, $s \in S$ is given by finite sum of elements of the form $fr(gs)$, $fr \in \text{Hom}_S(\Gamma, S)R$ and $gs \in \Gamma = \text{Hom}_R(T(S), R)S$. Now if ψ denotes the right R -module homomorphism $T(S) \rightarrow \text{Hom}_S(\text{Hom}_R(T(S), R)S, S)R$,

then for any $fr \in \text{Hom}_S(\text{Hom}_R(T(S), R)S, S)R$, there exists a $t \in T(S)$ such that $\psi(t)(gs) = fr(gs)$. But by definition,

$$\psi(t)(gs) = \eta_S^{-1} H(\varrho_t) H(g) \eta_S(s),$$

which by the definition of λ implies that $\lambda([t, gs]) = fr(gs)$. Hence $fr(gs) \in S$ corresponds to $[t, gs] \in [T(S), \Gamma]$ and so λ is an epimorphism and hence an isomorphism. This completes the proof.

Combining Theorems 2.3 and 2.5, we have the main theorem of this paper.

Theorem 2.6. *Let R and S be two rings satisfying the gamma conditions. Then R is Morita equivalent to S if and only if there exists a weakly semiprime gamma ring such that its right and left operator rings are isomorphic to R and S respectively.*

The first Wedderburn theorem for rings with unities can be interpreted in Morita language as "A ring is simple Artinian if and only if it is Morita equivalent to a division ring". As an application of Theorem 2.6 we have an extension of the Wedderburn theorem as follows.

Theorem 2.7. *Any ring satisfying the gamma conditions is Morita equivalent to a division ring if and only if it is simple and completely reducible.*

PROOF. Since a division ring has unity it satisfies the gamma conditions. So any ring Morita equivalent to it is Γ -context equivalent to it by Theorem 2.6. Suppose S is the ring Γ context equivalent to the division ring D and (A, Γ) be the gamma ring such that $L(A, \Gamma) \cong S$ and $R(A, \Gamma) \cong D$. If I is a nonzero left ideal of A , then $I^* = \{d \in D / Ad \subseteq I\}$ is a nonzero left ideal of D and hence $I^* = D$. It follows that $A = I$. Hence A is a faithful irreducible S -module. That is S is primitive. Now $S \subseteq \text{End}(A_D)$. Let $l = \sum_i [x_i, \alpha_i] \in S$. Then $l(a) = \sum_i [x_i, \alpha_i]a = \sum_i x_i[\alpha_i, a]$ for every $a \in A$. This implies that lA is finite dimensional. Hence S contains a nonzero linear transformation of finite rank [5] and since S is simple by Lemma 2.2, S must coincide with $\text{Soc } S$, the socle of S .

Conversely suppose S is simple and completely reducible and $S = \bigoplus I_\alpha$ where each I_α is a minimal left ideal of S . Then $I_\alpha = Se_\alpha$ for some idempotent $e_\alpha \in I_\alpha$. Se_α can be given a Γ -ring structure ($\Gamma = e_\alpha S$) and it can be easily checked that $R(Se_\alpha, e_\alpha S) \cong D \cong e_\alpha Se_\alpha$, a division ring.

It remains to show that $L(Se_\alpha, e_\alpha S) \cong S$. Define a map

$$\sigma: L(Se_\alpha, e_\alpha S) \rightarrow S \quad \text{by}$$

$$\sigma\left(\sum_i [l'_i e_\alpha, e_\alpha l_i]\right) = \sum_i l'_i e_\alpha l_i.$$

σ is clearly a ring monomorphism. We show σ is onto. Since Se_α and Se_β are minimal and S is simple we have $Se_\alpha \cong Se_\beta$ and given by $\varphi_\beta(xe_\alpha) = e_\beta$, $x \in S$. It follows that $e_\beta = xe_\alpha ye_\beta$ for some $y \in S$. Hence $le_\beta = le_\beta^2 = (e_\beta xe_\alpha)(e_\alpha ye_\beta)$. Hence σ is onto.

Se_α can be easily checked to be a weakly semiprime gamma ring. Se_α is a faithful S -module follows from the simplicity of S . So by Theorem 2.6, S is Morita equivalent to a division ring. This completes the proof.

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