Rees sublattices of a lattice

By JAROMÍR DUDA (Brno)

Rees congruences were introduced by D. REES [4] for semigroups. Recently R. F. TICHY [6] generalized this concept to arbitrary algebras as follows

Definition. Let B be a subalgebra of an algebra A, ω_A a diagonal on A. B is called a *Rees subalgebra* whenever $B\times B\cup\omega_A$ is a congruence on A. Any congruence of this form is called a *Rees congruence.*

Rees congruences on lattices were studied by G. SZÁSZ [5], namely an interesting characterization of Rees ideals was given in this paper. A characterization of Rees filters in implicative semilattices can be found in J. VARLET [7]. Since Varlet's result holds true for arbitrary lattices (see [2]), another description of Rees filters (Rees ideals) is obtained in this way. The present paper seeks to find suitable characterization theorems for arbitrary Rees sublattices of a given lattice. Further we study a remarkable relationship between Rees sublattices and arbitrary convex sublattices. Lattices having Rees sublattices only were already mentioned in the previous paper [1].

1. Characterization theorems

Theorem 1. *A convexs ublattice M of a lattice L is a Rees sublattice if, and only if one of the following conditions holds for any 3-element subset $\{a, b, x\}$ $a, b\in M$, $a<b$, $x\in L\setminus M$:*

(i) $\{a, b, x\}$ *is a chain,*

(ii) $\{a, b, x\}$ *generates a pentagon.*

PROOF. First suppose that M is a Rees sublattice of a lattice L. Choose arbitrary elements $a, b\in M$, $a<b$, and $x\in L\setminus M$. Assume further that $\{a, b, x\}$ is not a chain. Consider the following cases:

Case 1. Let $a\wedge x$, $b\vee x\in M$. Then the convexity of M implies $x\in M$, a contradiction.

Case 2. Suppose that $a\wedge x\in M$, $b\vee x\in L\setminus M$. Applying the elementary translation ι_x $(\iota_x(u)=u\vee x,\ u\in L)$ to the pair $\langle a\wedge x, b\rangle\in M\times M$ we get

$$\langle\iota_x(a\wedge x), \iota_x(b)\rangle = \langle(a\wedge x)\vee x, b\vee x\rangle = \langle x, b\vee x\rangle.$$

By hypothesis $\langle x, b\vee x\rangle\in\omega_L$, i.e. $x=b\vee x$ and so $b<x$, a contradiction.

Case 3. Analogously we find that the assumption $a\wedge x\in L\setminus M$, $b\vee x\in M$ is impossible.

Case 4. Suppose that $a\wedge x, b\vee x\in L\setminus M$. Apply the elementary translation ι_x to the pair $\langle a, b\rangle\in M\times M$. Then $\langle \iota_x(a), \iota_x(b)\rangle=\langle a\vee x, b\vee x\rangle\in\omega_L$, i.e. $a\vee x=b\vee x$. Similarly the equality $a\wedge x=b\wedge x$ can be proved. Hence the elements $a<b, x, a\vee x=$ $=b\vee x$, and $a\wedge x=b\wedge x$ form a pentagon, as claimed.

Conversely assume that arbitrarily chosen elements a, b, x, $a, b\in M$, $a<b$, $x\in L\setminus M$, satisfy one of the conditions (i), (ii) of our theorem. It suffices to verify that the equivalence relation $M\times M\cup\omega_L$ on L is compatible with the elementary translations ι_x and μ_x $(\mu_x(u)=u\wedge x, u\in L)$ for any $x\in L\setminus M$. To do this take elements $c, d\in M$, $c\neq d$. Denote by $a=c\wedge d$ and $b=c\vee d$. Clearly than $a<b$. By hypothesis two possibilities may occur:

Case 1. If $\{a, b, x\}$ is a chain, say $a<b<x$, then

$$\langle \iota_x(c), \iota_x(d)\rangle = \langle c\wedge x, d\wedge x\rangle = \langle x, x\rangle\in\omega_L \subseteqq M\times M\cup\omega_L$$

and

$$\langle \mu_x(c), \mu_x(d)\rangle = \langle c\wedge x, d\wedge x\rangle = \langle c, d\rangle\in M\times M \subseteqq M\times M\cup\omega_L.$$

Evidently the same conclusion holds for $x<a<b$.

Case 2. Suppose that $\{a, b, x\}$ generates the pentagon

$$\{a, b, x, a\vee x = b\vee x, a\wedge x = b\wedge x\}.$$

Then we have also $c\vee x=d\vee x$ and $c\wedge x=d\wedge x$, i.e.

$$\langle \iota_x(c), \iota_x(d)\rangle\in\omega_L \subseteqq M\times M\cup\omega_L$$

and

$$\langle \mu_x(c), \mu_x(d)\rangle\in\omega_L \subseteqq M\times M\cup\omega_L.$$

The proof is complete.

Making use of Theorem 1 another characterization of Rees sublattices can be obtained. First we need some preliminaries.

Having two nonvoid subsets M, N of a lattice L denote by

$$M\vee N = \{m\vee n;\ m\in M, n\in N\}.$$

and, dually,

$$M\wedge N = \{m\wedge n;\ m\in M, n\in N\}$$

Lemma 1. *Let R be a nontrivial Rees sublattice of a lattice L, $x\in L\setminus R$, and $d<x<c$. Then*

(a) $R\vee\{x\}=\{x\}$ *if and only if* $r\vee x=x$ *for some element* $r\in R$;

(b) $R\wedge\{x\}=\{x\}$ *if and only if* $r\wedge x=x$ *for some element* $r\in R$;

(c) $R\vee\{x\}=\{c\}$ *and* $R\wedge\{x\}=\{d\}$ *if and only if* $r\vee x=c$ *and* $r\wedge x=d$ *for some element* $r\in R$.

PROOF. Assertions (a), (b) are evident.

(c) Let $s\in R$, $s\neq r$. Then $a<b$ for $a=s\wedge r$ and $b=s\vee r$. Apply Theorem 1 to the subset $\{a, b, x\}$: If $a<b<x$ then $r<x$, i.e. $c=r\vee x=x$, a contradiction.

Similarly the case $x<a<b$ is impossible. Hence $a\vee x=b\vee x$ and $a\wedge x=b\wedge x$ hold. Consequently $s\vee x=r\vee x=c$ and $s\wedge x=r\wedge x=d$ for any $s\in R$.

Theorem 2. *A nontrivial convex sublattice M of a lattice L is a Rees sublattice if and only if one of the following conditions holds for any element $x\in L\setminus M$:*

(i) $M\vee\{x\}=\{x\}$,

(ii) $M\wedge\{x\}=\{x\}$,

(iii) *The subsets $M\vee\{x\}$ and $M\wedge\{x\}$ are singletons.*

PROOF. First suppose that M is a Rees sublattice of L. Choose elements $m\in M$ and $x\in L\setminus M$. If m and x are comparable apply Lemma 1 (a), (b). In the opposite case apply Lemma 1 (c).

The converse implication is evident.

Corollary 1. *A nontrivial convex sublattice M of a modular lattice L is a Rees sublattice if and only if each element from M is comparable with every element from $L\setminus M$.*

PROOF. Apply Theorem 1 and the Dedekind criterion for modular lattices, see e.g. [3].

As noted above, Rees ideals and Rees filters were already studied in [5], [7], and [2]. Our next theorem summarizes the former results.

Theorem 3. *Let I be an ideal of a lattice L. The following conditions are equivalent:*

(1) *I is a Rees ideal;*

(2) *$L\setminus I$ is a set of upper bounds of I;*

(3) *I is a nodal ideal, i.e. I is comparable with any other ideal in I.*

PROOF. The equivalence $(1)\Leftrightarrow(2)$ is due to G. Szász [5], $(1)\Leftrightarrow(3)$ can be found in [2]. For the sake of completeness we present here a short proof of Theorem 3:

$(1)\Rightarrow(2)$. Take $x\in L\setminus I$. Then $i\wedge x\in I$ and $i\wedge x<x$ for arbitrary $i\in I$. Lemma 1 (a) implies (2).

$(2)\Rightarrow(3)$. Immediate.

$(3)\Rightarrow(1)$. Choose an element $x\in L\setminus I$ and consider the principal ideal $(x]$. By hypothesis (3) the inclusion $(x]\supset I$ holds, i.e. we have $I\vee\{x\}=\{x\}$. Theorem 2 completes the proof.

The relationship among Rees sublattices, Rees ideals and Rees filters can be expressed as follows

Corollary 2. *A convex sublattice M of a lattice L is a Rees sublattice whenever the ideal $(M]$ and the filter $[M)$ are Rees.*

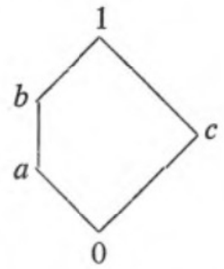
If M is a nontrivial Rees sublattice of a modular lattice L then the ideal $(M]$ and the filter $[M)$ are Rees.

PROOF. The first assertion follows from the evident inclusions $M\times M\cup\omega_L\subseteq$ $\subseteq\Theta(M)$ and

$$\Theta(M)\subseteq\Theta((M])\cap\Theta([M))=M\times M\cup\omega_L.$$

The second part of our corollary is a direct consequence of Corollary 1 and Theorem 3(2).

Remark 1. The assumption of modularity in the second part of Corollary 2 is essential and hence it cannot be omitted. Counterexample: Let L be a pentagon:



Then the interval $[a, b]$ is Rees sublattice of L but neither $(b]$ nor $[a)$ have this property.

For classes of lattices closed under the formation of sublattices a sufficient condition for modularity can be obtained from Corollary 2. We formulate this fact for varieties of lattices.

Corollary 3. *Let $\mathscr{V}$ be a variety of lattices. The following conditions are equivalent:*

(1) *Any nontrivial convex sublattice M of a lattice $L\in\mathscr{V}$ is a Rees sublattice if and only if the ideal $(M]$ and the filter $[M)$ are Rees.*

(2) *$\mathscr{V}$ is a variety of modular lattices.*

PROOF. (2)⇒(1) follows from Corollary 2. To prove the converse implication suppose that V is not modular. Then V contains a pentagon, by Dedekind criterion. Consequently (1) does not hold, see Remark 1.

2. Rees sublattices and arbitrary convex sublattices

Theorem 3 of the preceding section shows that a Rees ideal (Rees filter) is in a special position with respect to any other ideal (filter, respectively). Now we turn our attention to the relationship between a Rees sublattice and an arbitrary convex sublattice. For nondisjoint pairs of such sublattices holds

Theorem 4. *Let R be a Rees sublattice of a lattice L. Then for any nondisjoint convex sublattice M of L one of the following possibilities occurs:*

(i) $M\subseteq R$,

(ii) $R\subset M$,

(iii) *Elements from $M\setminus R$ are upper bounds of R,*

(iv) *Elements from $M\setminus R$ are lower bounds of R.*

Furthermore, the set union $R\cup M$ is a convex sublattice of L in any case.

PROOF. If $M \setminus R = \emptyset$ then $M \subseteq R$, i.e. (i) holds.

In the opposite case we apply Theorem 2 to the elements of $M \setminus R$: If $M \setminus R$ contains an element m such that $R \vee \{m\}$ and $R \wedge \{m\}$ are singletons then $R \vee \{m\} = \{a \vee m\}$ and $R \wedge \{m\} = \{a \wedge m\}$ for an element $a \in R \cap M$. Since $a \wedge m$, $a \vee m \in M$, the convexity of M implies $R \subset M$, i.e. the case (ii) occurs.

If $M \setminus R$ contains both lower and upper bounds of R then the case (ii) is again obtained.

Finally if $M \setminus R$ consists of upper (lower) bounds of R only, then the case (iii) ((iv), respectively) follows.

It remains to prove that $R \cup M$ is a convex sublattice of L. Without loss of generality suppose that (iii) holds. Take elements $m \in M \setminus R$ and $r \in R$. Then $r < m$. Consider an arbitrary element $t \in [r, m]$ such that $t \notin R$. Since R is Rees Lemma 1 yields that t is an upper bound of R, namely $a < t$ for any element $a \in R \cap M$. In this way we find that $t \in [a, m] \subseteq M \subseteq R \cup M$ which was to be proved.

For modular lattices we can state

Theorem 5. *Let R be a nontrivial Rees sublattice of a modular lattice L. Then every convex sublattice M of L disjoint from R consits of upper (lower) bounds of R only.*

PROOF. Apply Corollary 1.

3. Pairs of Rees sublattices

Finally only pairs of Rees sublattices are considered. For nondisjoint pairs of Rees sublattices holds

Theorem 6. *Let R_1, R_2 be nondisjoint Rees sublattices of a lattice L. Then $R_1 \cup R_2$ is a Rees sublattice of L.*

PROOF. As was already proved in Theorem 4, $R_1 \cup R_2$ is a convex sublattice of L. Take elements $a \in R_1 \cup R_2$ and $x \in L \setminus (R_1 \cup R_2)$. Since R_1, R_2 are Rees sublattices Lemma 1 yields the possibilities:

(i) $(R_1 \cup R_2) \vee \{x\} = \{x\}$,

(ii) $(R_1 \cup R_2) \wedge \{x\} = \{x\}$,

(iii) $(R_1 \cup R_2) \vee \{x\}$ and $(R_1 \cup R_2) \wedge \{x\}$ are singletons.

Theorem 2 completes the proof.

It remains clarify the relationship between two disjoint Rees sublattices.

Theorem 7. *Let R_1, R_2 be disjoint Rees sublattices of a lattice L. Then one of the following possibilities occurs:*

(i) *Elements from R_2 are upper bounds of R_1,*

(ii) *Elements from R_2 are lower bounds of R_1,*

(iii) *The subsets $R_1 \vee R_2$ and $R_1 \wedge R_2$ are singletons.*

PROOF. First suppose that R_1 (R_2) contains an element x_1 (x_2) which is an upper or lower bound of R_2 (R_1, respectively). For instance let $x_1 \in R_1$ be a lower bound of R_2. Then $x_1 < x_2$ hold for every $x_2 \in R_2$. Since R_1 is a Rees sublattice Lemma 1 (a) yields that x_2 is an upper bound of R_1 for any $x_2 \in R_2$. Hence (i) holds. Similarly (ii) can be obtained.

In the opposite case Theorem 2 states that subsets $R_1\vee\{x_2\}$, $R_1\wedge\{x_2\}$ and $R_2\vee\{x_1\}$, $R_2\wedge\{x_1\}$ are singletons for any $x_1\in R_1$, $x_2\in R_2$. Fix $a_1\in R_1$, $a_2\in R_2$. Then

$$R_1\vee R_2=\bigcup_{x_2\in R_2} R_1\vee\{x_2\}=\bigcup_{x_2\in R_2}\{a_1\vee x_2\}=\{a_1\}\vee R_2=\{a_1\vee a_2\},$$

and

$$R_1\wedge R_2=\bigcup_{x_2\in R_2} R_1\wedge\{x_2\}=\bigcup_{x_2\in R_2}\{a_1\wedge x_2\}=\{a_1\}\wedge R_2=\{a_1\wedge a_2\}$$

proving (iii).

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Author's address:
JAROMÍR DUDA
616 00 BRNO 16, KROFTOVA 21
CZECHOSLOVAKIA

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