

Recurrent curvature tensors in Finsler spaces with generalised connection

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§ 1. *Introduction.* There are several papers in which the recurrent Finsler spaces are examined. We note that recurrency in these papers is defined in different manner. There exist a great number of Finsler spaces in which metric connection may be defined. This possibility depends on the metric function $F(x, \dot{x})$. Some authors as R. B. MISHRA in 1 S. P. SINGH in [2] and others have considered some classes of Finsler spaces supplied with the metric connections which satisfy certain recurrency condition for a curvature tensor, as for example

$$R_{\alpha\beta\gamma\delta|\kappa} = A_{\kappa} R_{\alpha\beta\gamma\delta} \quad \text{or} \quad K_{\alpha\beta\gamma\delta|\kappa|e=\alpha_{\kappa\tau}} K_{\alpha\beta\gamma\delta}.$$

The above equations can be considered as systems of partial differential equations for the metric function $F(x, \dot{x})$, because the tensors R and K are defined by the connection coefficients $\Gamma_{\alpha\beta}^{\gamma}$, and they are functions of the metric tensor g which is determined by the $F(x, \dot{x})$ in the form of (2.1). Only those Finsler spaces which are constructed by the metric function $F(x, \dot{x})$ satisfying one of the above equations allow the recurrent curvature tensor.

In the present paper we shall examine recurrent curvature tensors in certain generalised Finsler spaces. A. MOÓR in [3] gave a generalisation of Finsler spaces in such a way that the metric tensor is not covariant constant, but only recurrent i.e. (2.6c) and (2.6d) are satisfied. For $\lambda_{\delta}=0$ and $\mu_{\delta}=0$ we obtain the classical Finsler spaces. In this paper we shall consider such kind of generalised Finsler space $\bar{F}_n$, which coincides with Moór's generalisation in the case of $\mu_{\delta}=0$. The difference between the Moór's and our cases arises from the fact that $A_{\alpha\beta\gamma}$ in [3] is symmetric in its first two indices, but here in the first and last one. The purpose of this paper is to examine such classes of $\bar{F}_n$ in which one of the curvature tensors is recurrent with respect to $|\kappa$ or $|\kappa$ defined by (2.4) and (2.5). The explicit solution for $F(x, \dot{x})$ satisfying the recurrency condition for one of the curvature tensors K, R, P or S will not be given, we only obtain some conditions for the vectors λ, μ and the vectors of recurrency.

§ 2. *Some definitions and notations.* We use the symbols and notations of [4] and [5] without any explanation. Let $F(x, \dot{x})$ be the metric function in the space $\bar{F}_n$, then the metric tensor g is defined by

$$(2.1) \quad g_{\alpha\beta}(x, \dot{x}) = 2^{-1} \partial_{\alpha} \partial_{\beta} F^2(x, \dot{x})$$

as usual. If $\xi^{\alpha}(x, \dot{x})$ are coordinates of a vectorfield homogeneous of degree zero

in $\dot{x}$, then

$$(2.2) \quad D\xi^\alpha = d\xi^\alpha + \Gamma_{\beta\gamma}^{*\alpha} \xi^\beta dx^\gamma + A_{\beta\gamma}^\alpha \xi^\beta Dl^\gamma,$$

$$Dl^\gamma = dl^\gamma + \Gamma_{0\gamma}^{*\alpha} dx^\gamma + A_{0\gamma}^\alpha Dl^\gamma,$$

$$(2.3) \quad D\xi^\alpha = \xi_{|\beta}^\alpha dx^\beta + \xi^\alpha|_\beta Dl^\beta,$$

$$(2.4) \quad \xi_{|\beta}^\alpha = \partial_\beta \xi^\alpha - \dot{\partial}_\alpha \xi^\alpha \Gamma_{\beta}^{*\alpha} + \Gamma_{\alpha\beta}^{*\alpha} \xi^\alpha,$$

$$(2.5) \quad \xi^\alpha|_\beta = F \partial_\alpha \xi^\alpha (\delta_\beta^\alpha - A_{0\beta}^\alpha) + A_{\alpha\beta}^\alpha \xi^\alpha.$$

The connection coefficients in $\bar{F}_n$ are given in [4]. They satisfy the relations

$$(2.6) \quad \begin{aligned} &\text{a) } \Gamma_{\beta\gamma}^{*\alpha} = \Gamma_{\gamma\beta}^{*\alpha} & \text{b) } A_{\beta\gamma}^\alpha = A_{\gamma\beta}^\alpha \\ &\text{c) } g_{\alpha\beta|\delta} = \lambda_\delta g_{\alpha\beta} & \text{d) } g_{\alpha\beta|\delta} = \mu_\delta g_{\alpha\beta} \end{aligned}$$

and have the form

$$(2.7) \quad \begin{aligned} &\text{a) } \Gamma_{\beta\gamma}^{*\alpha} = \tilde{\Gamma}_{\beta\gamma}^{*\alpha} + T_{\beta\gamma}^\alpha(g, \lambda) \\ &\text{b) } A_{\beta\gamma}^\alpha = \tilde{A}_{\beta\gamma}^\alpha + Q_{\beta\gamma}^\alpha(g, \mu) \end{aligned}$$

where $\tilde{\Gamma}_{\beta\gamma}^{*\alpha}$ and $\tilde{A}_{\beta\gamma}^\alpha$ are connection coefficients of the Cartan connection deduced from $F(x, \dot{x})$, i.e. if $\lambda_\delta=0, \mu_\delta=0$. This space will be referred to as "ordinary Finsler space". $T_{\beta\gamma}^\alpha(g, \lambda), Q_{\beta\gamma}^\alpha(g, \mu)$ are tensors which vanish for $\lambda_\delta=0$ and $\mu_\delta=0$ respectively. We shall use the relations [4]:

$$(2.8) \quad \begin{aligned} &\text{a) } l_{\beta|\alpha} = 2^{-1} l_\beta \lambda_\alpha & \text{b) } l_{|\alpha}^\beta = -2^{-1} l^\beta \lambda_\alpha \\ &\text{c) } l_{\beta|\alpha} = 2^{-1} l_\beta \mu_\alpha + h_{\beta\alpha} & \text{d) } l_{|\alpha}^\beta = -2^{-1} l^\beta \mu_\alpha + h_{\alpha}^\beta, \end{aligned}$$

where

$$h_{\beta\alpha} = g_{\beta\alpha} - l_\beta l_\alpha \quad h_{\alpha}^\beta = \delta_{\alpha}^\beta - l^\beta l_\alpha.$$

The curvature tensors in $\bar{F}_n$ are defined by

$$(2.9) \quad 2^{-1} K_{\alpha\gamma\delta}^\beta = \partial_{[\delta} \Gamma_{|\alpha|\gamma]}^{*\beta} - \dot{\partial}_i \Gamma_{\alpha[\gamma}^{*\beta} \Gamma_{\delta]}^{*i} + \Gamma_{\alpha[\gamma}^{*\alpha} \Gamma_{|\delta]}^{*\beta},$$

$$(2.10) \quad R_{\alpha\gamma\delta}^\beta = K_{\alpha\gamma\delta}^\beta + A_{\alpha\delta}^\beta K_{0\gamma\delta}^\alpha,$$

$$(2.11) \quad P_{\alpha\gamma\delta}^\beta = F \dot{\partial}_\alpha \Gamma_{\alpha\gamma}^{*\beta} (\delta_\delta^\alpha - A_{0\delta}^\alpha) - A_{\alpha\delta|\gamma}^\beta + A_{\alpha\delta}^\beta \dot{x}^\alpha \dot{\partial}_\gamma \Gamma_{\alpha\delta}^{*i},$$

$$(2.12) \quad 2^{-1} S_{\alpha\gamma\delta}^\beta = F \dot{\partial}_\alpha A_{\alpha\gamma}^\beta (\delta_\delta^\alpha - A_{0\delta}^\alpha) + A_{\alpha\gamma}^\alpha A_{|\delta]}^\beta.$$

These tensors are formed with connection coefficients of the recurrent Finsler space $\bar{F}_n$. In case of an ordinary Finsler space ($\lambda_\gamma=0, \mu_\gamma=0$) these connection coefficients reduce to the corresponding connection coefficients of the not recurrent Finsler space and the above defined curvature tensors become the wellknown curvature tensors in the ordinary Finsler space (where $A_{0\delta}^\alpha=0$). The tensors defined

by (2.9)—(2.12) satisfy the following relations [5]

$$(2.13) \quad -K_{\alpha\beta\gamma\delta} - K_{\beta\alpha\gamma\delta} - F\dot{\partial}_\kappa g_{\alpha\beta} K_{0\gamma\delta}^\kappa = 2\lambda_{[\gamma|\delta]} g_{\alpha\beta},$$

$$(2.14) \quad -P_{\alpha\beta\gamma\delta} - P_{\beta\alpha\gamma\delta} - g_{\alpha\beta} \lambda_\kappa A_{\gamma\delta}^\kappa + \\ + F\dot{\partial}_\kappa g_{\alpha\beta} [A_{0\delta|\gamma}^\kappa - (\dot{x}^\theta \dot{\partial}_\theta \Gamma_{\delta\gamma}^{*\kappa} + 2^{-1} \lambda_\gamma \delta_i^\kappa) (\delta_i^\delta - A_{0\delta}^i)] = \\ = (\lambda_\gamma|_\delta - \mu_\delta|_\gamma) g_{\alpha\beta},$$

$$(2.15) \quad -S_{\alpha\beta\gamma\delta} - S_{\beta\alpha\gamma\delta} - 2F^2 \dot{\partial}_\kappa g_{\alpha\beta} (\dot{\partial}_{[\delta} A_{0|\gamma]}^\kappa - \dot{\partial}_i A_{0[\gamma}^\kappa A_{|\delta]}^i) = 2\mu_{[\gamma|\delta]} g_{\alpha\beta}.$$

From the above equations using (2.1) after contraction with l^α we obtain

$$(2.16) \quad -K_{0\beta\gamma\delta} - K_{\beta 0\gamma\delta} = (\lambda_\gamma|_\delta - \lambda_\delta|_\gamma) l_\beta,$$

$$(2.17) \quad -P_{0\beta\gamma\delta} - P_{\beta 0\gamma\delta} = l_\beta \lambda_\epsilon A_{\gamma\delta}^\epsilon + (\lambda_\gamma|_\delta - \mu_\delta|_\gamma) l_\beta,$$

$$(2.18) \quad -S_{0\beta\gamma\delta} - S_{\beta 0\gamma\delta} = (\mu_\gamma|_\delta - \mu_\delta|_\gamma) l_\beta.$$

The contraction of (2.16)—(2.18) with l^β yields:

$$(2.19) \quad -2K_{00\gamma\delta} = \lambda_\gamma|_\delta - \lambda_\delta|_\gamma,$$

$$(2.20) \quad -2P_{00\gamma\delta} = \lambda_\epsilon|_\delta A_{\gamma\delta}^\epsilon + \lambda_\gamma|_\delta - \mu_\delta|_\gamma,$$

$$(2.21) \quad -2S_{00\gamma\delta} = \mu_\gamma|_\delta - \mu_\delta|_\gamma.$$

§ 3. The $|\kappa$ recurrent curvature tensors in $\bar{F}_n$.

Definition 3.1. If for some tensor $T:::$ in $\bar{F}_n$ exists a vectorfield $a_\kappa(x, \dot{x})$ homogeneous of degree zero in $\dot{x}$, such that

$$(3.1) \quad T:::|_\kappa = a_\kappa T:::$$

then the tensor $T:::$ is $|\kappa$ recurrent with vector of recurrency a_κ .

Lemma 3.1. If $T_{\alpha\gamma\delta}^\beta$ is a tensorfield in $\bar{F}_n$ homogeneous of degree zero in $\dot{x}$ and if it is $|\kappa$ recurrent with vector of recurrency t_κ i.e.

$$(3.2) \quad T_{\alpha\gamma\delta|_\kappa}^\beta = t_\kappa T_{\alpha\gamma\delta}^\beta,$$

then

$$(3.3) \quad T_{\alpha\beta\gamma\delta|_\kappa} = (t_\kappa + \lambda_\kappa) T_{\alpha\beta\gamma\delta},$$

$$(3.4) \quad T_{0\gamma\delta|_\kappa}^\beta = (t_\kappa - 2^{-1} \lambda_\kappa) T_{0\gamma\delta}^\beta,$$

$$(3.5) \quad T_{0\beta\gamma\delta|_\kappa} = (t_\kappa + 2^{-1} \lambda_\kappa) T_{0\beta\gamma\delta},$$

$$(3.6) \quad T_{\beta 0\gamma\delta|_\kappa} = (t_\kappa + 2^{-1} \lambda_\kappa) T_{\beta 0\gamma\delta},$$

$$(3.7) \quad T_{00\gamma\delta|_\kappa} = t_\kappa T_{00\gamma\delta}.$$

PROOF. From (3.2), (2.6c) and

$$(3.8) \quad T_{\alpha\beta\gamma\delta} = g_{\epsilon\beta} T_{\alpha\beta\gamma}^\epsilon$$

we obtain (3.3). From (3.2), (2.8b) and

$$(3.9) \quad T_{0\gamma\delta|\kappa}^\beta = T_{\alpha\gamma\delta|\kappa}^\beta l^\alpha + T_{\alpha\gamma\delta}^\beta l_{|\kappa}^\alpha$$

follows (3.4). From (3.3) and (2.8b) follow (3.5), (3.6) and (3.7).

Theorem 3.1. *If $K_{\alpha\gamma\delta}^\beta$ in $\bar{F}_n$ is $|\kappa$ recurrent, then also $\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}$ is $|\kappa$ recurrent with the same vector of recurrency k_κ .*

PROOF. From the condition of theorem we have

$$(3.10) \quad K_{\alpha\gamma\delta|\kappa}^\beta = k_\kappa K_{\alpha\gamma\delta}^\beta.$$

Using Lemma 3.1 we obtain

$$K_{0\beta\gamma\delta|\kappa} = (k_\kappa + 2^{-1}\lambda_\kappa)K_{0\beta\gamma\delta}, \quad K_{\beta 0\gamma\delta} = (k_\kappa + 2^{-1}\lambda_\kappa)K_{\beta 0\gamma\delta}.$$

The sum of the above two equations gives

$$(3.11) \quad (K_{0\beta\gamma\delta} + K_{\beta 0\gamma\delta})_{|\kappa} = (k_\kappa + 2^{-1}\lambda_\kappa)(K_{0\beta\gamma\delta} + K_{\beta 0\gamma\delta}).$$

Using (2.16) we obtain from (3.11)

$$[(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})l_\beta]_{|\kappa} = (k_\kappa + 2^{-1}\lambda_\kappa)(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})l_\beta.$$

Using (2.8a) we get

$$(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})_{|\kappa} l_\beta = k_\kappa (\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})l_\beta.$$

Contraction of this equation with l^β gives

$$(3.12) \quad (\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})_{|\kappa} = k_\kappa (\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}).$$

A shorter proof is the following: (3.10) $\wedge$ (3.7) $\Rightarrow$

$$(3.13) \quad K_{00\gamma\delta|\kappa} = k_\kappa K_{00\gamma\delta},$$

$$(3.13) \quad \wedge (2.19) \Rightarrow (3.12).$$

Theorem 3.2. *If $P_{\alpha\gamma\delta}^\beta$ is $|\kappa$ recurrent, then also $\lambda_{\gamma|\delta} - \mu_{\delta|\gamma} + \lambda_\epsilon A_{\gamma\delta}^\epsilon$ is $|\kappa$ recurrent with the same vector of recurrency p_κ .*

PROOF. From the condition of the theorem we have

$$(3.14) \quad P_{\alpha\gamma\delta|\kappa}^\beta = p_\kappa P_{\alpha\gamma\delta}^\beta.$$

Substituting in Lemma 3.1 $T_{\alpha'\gamma\delta}^\beta$ and t_κ by $P_{\alpha\gamma\delta}^\beta$ and p_κ , we obtain

$$(3.15) \quad (P_{0\beta\gamma\delta} + P_{\beta 0\gamma\delta})_{|\kappa} = (p_\kappa + 2^{-1}\lambda_\kappa)(P_{0\beta\gamma\delta} + P_{\beta 0\gamma\delta}).$$

Substituting (2.17) into (3.15) and using (2.8a) we get

$$(3.16) \quad (\lambda_{\gamma|\delta} - \mu_{\delta|\gamma} + \lambda_\epsilon A_{\gamma\delta}^\epsilon)_{|\kappa} l_\beta = p_\kappa (\lambda_{\gamma|\delta} - \mu_{\delta|\gamma} + \lambda_\epsilon A_{\gamma\delta}^\epsilon)l_\beta.$$

Then contraction with l^β proves the theorem, i.e.

$$(\lambda_{\gamma|\delta} - \mu_{\delta|\gamma} + \lambda_\epsilon A_{\gamma\delta}^\epsilon)_{|\kappa} = p_\kappa (\lambda_{\gamma|\delta} - \mu_{\delta|\gamma} + \lambda_\epsilon A_{\gamma\delta}^\epsilon).$$

Theorem 3.3. *If $S_{\alpha\gamma\delta}^\beta$ is $|\kappa$ recurrent, then also $\mu_\gamma|_\delta - \mu_\delta|_\gamma$ is $|\kappa$ recurrent with the same vector of recurrency s_κ .*

PROOF. From the condition of the theorem we have

$$(3.17) \quad S_{\alpha\gamma\delta}^\beta|_\kappa = s_\kappa S_{\alpha\gamma\delta}^\beta.$$

Using Lemma 3.1, and substituting $T_{\alpha\gamma\delta}^\beta$ and t_κ with $S_{\alpha\gamma\delta}^\beta$ and s_κ respectively, we obtain

$$(3.18) \quad (S_{\beta 0\gamma\delta} + S_{0\beta\gamma\delta})|_\kappa = (s_\kappa + 2^{-1}\lambda_\kappa)(S_{\beta 0\gamma\delta} + S_{0\beta\gamma\delta}).$$

Substituting (2.18) into (3.18), and using (2.8a) we get

$$(3.19) \quad (\mu_\gamma|_\delta - \mu_\delta|_\gamma)|_\kappa l_\beta = s_\kappa(\mu_\gamma|_\delta - \mu_\delta|_\gamma)l_\beta.$$

A contraction of (3.19) with l^β proves the theorem, i.e. (3.17) $\Rightarrow$

$$(3.20) \quad (\mu_\gamma|_\delta - \mu_\delta|_\gamma)|_\kappa = s_\kappa(\mu_\gamma|_\delta - \mu_\delta|_\gamma)$$

§ 4. *The $|\kappa$ recurrent curvature tensors in $\bar{F}_n$.*

Definition 4.1. If for some tensor $T:::$ in $\bar{F}_n$ exists a vectorfield $\bar{a}_\kappa(x, \dot{x})$ homogeneous of degree zero in $\dot{x}$, such that

$$(4.1) \quad (T:::)|_\kappa = \bar{a}_\kappa T:::,$$

then the tensor $T:::$ is $|\kappa$ recurrent with the vector of recurrency $\bar{a}_\kappa$.

Lemma 4.1. *If $T_{\alpha\gamma\delta}^\beta$ is a tensorfield in $\bar{F}_n$ homogeneous of degree zero in $\dot{x}$, and if it is $|\kappa$ recurrent with vector of recurrency $\bar{i}_\kappa$, i.e.*

$$(4.2) \quad T_{\alpha\gamma\delta}^\beta|_\kappa = \bar{i}_\kappa T_{\alpha\gamma\delta}^\beta,$$

then

$$(4.3) \quad T_{\alpha\beta\gamma\delta}|_\kappa = (\bar{i}_\kappa + \mu_\kappa)T_{\alpha\beta\gamma\delta},$$

$$(4.4) \quad T_{0\beta\gamma\delta}|_\kappa = (\bar{i}_\kappa + 2^{-1}\mu_\kappa - l_\kappa)T_{0\beta\gamma\delta} + T_{\kappa\beta\gamma\delta},$$

$$(4.5) \quad T_{\beta 0\gamma\delta}|_\kappa = (\bar{i}_\kappa + 2^{-1}\mu_\kappa - l_\kappa)T_{\beta 0\gamma\delta} + T_{\beta\kappa\gamma\delta},$$

$$(4.6) \quad T_{00\gamma\delta}|_\kappa = (\bar{i}_\kappa - 2l_\kappa)T_{00\gamma\delta} + T_{\kappa 0\gamma\delta} + T_{0\kappa\gamma\delta}.$$

PROOF. From (4.2) and (2.6d) follows (4.3). From (4.3) and (2.8d) follow (4.4) and (4.5). From (4.4), or (4.5) and (2.8d) follows (4.6).

Lemma 4.2. *If $T_{\alpha\gamma\delta}^\beta$ is $|\kappa$ recurrent in $\bar{F}_n$, and if exist vectorfields $w_\kappa(x, \dot{x})$ and $w'_\kappa(x, \dot{x})$ homogeneous of degree zero in $\dot{x}$ such that*

$$(4.7) \quad T_{\kappa\beta\gamma\delta} = w_\kappa T_{0\beta\gamma\delta},$$

$$(4.8) \quad T_{\beta\kappa\gamma\delta} = w'_\kappa T_{\beta 0\gamma\delta},$$

then $T_{0\beta\gamma\delta}$ and $T_{\beta 0\gamma\delta}$ are $|\kappa$ recurrent and we have

$$(4.9) \quad T_{0\beta\gamma\delta}|_{\kappa} = (\bar{l}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa} + w_{\kappa})T_{0\beta\gamma\delta},$$

$$(4.10) \quad T_{\beta 0\gamma\delta}|_{\kappa} = (\bar{l}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa} + w'_{\kappa})T_{\beta 0\gamma\delta},$$

$$(4.11) \quad T_{00\gamma\delta}|_{\kappa} = (\bar{l}_{\kappa} - 2l_{\kappa} + w_{\kappa} + w'_{\kappa})T_{00\gamma\delta},$$

$$(4.12) \quad T_{\kappa 0\gamma\delta} + T_{0\kappa\gamma\delta} = (w_{\kappa} + w'_{\kappa})T_{00\gamma\delta}.$$

PROOF. Substituting (4.7) and (4.8) into (4.4) and (4.5) we obtain (4.9) and (4.10). Contraction of (4.7) and (4.8) with l^{β} gives

$$(4.13) \quad T_{\kappa 0\gamma\delta} = w_{\kappa}T_{00\gamma\delta},$$

$$(4.14) \quad T_{0\kappa\gamma\delta} = w'_{\kappa}T_{00\gamma\delta}.$$

Substituting (4.13) and (4.14) into (4.6) we obtain (4.11). The sum of (4.13) and (4.14) gives (4.12).

Lemma 4.3. If $K_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent, then also $K_{00\gamma\delta}$ is $|\kappa$ recurrent with the same vector of recurrency $\bar{k}_{\kappa}$.

PROOF. Since

$$(4.15) \quad K_{\alpha\gamma\delta}^{\beta}|_{\kappa} = \bar{k}_{\kappa}K_{\alpha\gamma\delta}^{\beta}$$

it follows from Lemma 4.1 that (4.2)–(4.6) remain valid if we write K and $\bar{k}$ in place of T and $\bar{l}$ respectively. Now (4.4) has the form

$$K_{0\beta\gamma\delta}|_{\kappa} = (\bar{k}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa})K_{0\beta\gamma\delta} + K_{\kappa\beta\gamma\delta}.$$

Contracting this equation with l^{β} , and using (2.8d), we obtain

$$K_{00\gamma\delta}|_{\kappa} = (\bar{k}_{\kappa} - 2l_{\kappa})K_{00\gamma\delta} + (K_{\kappa 0\gamma\delta} + K_{0\kappa\gamma\delta}).$$

With respect to (2.16) and (2.19) we get

$$(4.16) \quad K_{00\gamma\delta}|_{\kappa} = \bar{k}_{\kappa}K_{00\gamma\delta}.$$

So we proved that (4.15) $\Rightarrow$ (4.16).

Lemma 4.4. If $K_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent with vector of recurrency $\bar{k}_{\kappa}$, and there exist vectorfields ξ_{κ} and ξ'_{κ} such that

$$(4.17) \quad K_{\kappa\beta\gamma\delta} = \xi_{\kappa}K_{0\beta\gamma\delta},$$

$$(4.18) \quad K_{\beta\kappa\gamma\delta} = \xi'_{\kappa}K_{\beta 0\gamma\delta},$$

then $K_{0\beta\gamma\delta}$ and $K_{\beta 0\gamma\delta}$ are $|\kappa$ recurrent:

$$(4.19) \quad K_{0\beta\gamma\delta}|_{\kappa} = (\bar{k}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa} + \xi_{\kappa})K_{0\beta\gamma\delta},$$

$$(4.20) \quad K_{\beta 0\gamma\delta}|_{\kappa} = (\bar{k}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa} + \xi'_{\kappa})K_{\beta 0\gamma\delta},$$

and

$$(4.21) \quad \xi_{\kappa} + \xi'_{\kappa} = 2l_{\kappa}.$$

PROOF. From Lemma 4.2, (4.17) and (4.18) follows that (4.7)–(4.14) are valid provided we substitute T , $\bar{i}$, w and w' with K , $\bar{k}$, ξ and ξ' respectively. (4.19) and (4.20) are the new form of (4.10) and (4.11). (4.12) has now the form:

$$K_{\kappa 0 \gamma \delta} + K_{0 \kappa \gamma \delta} = (\xi_{\kappa} + \xi'_{\kappa}) K_{00 \gamma \delta}.$$

Using (2.16) and (2.19) we obtain

$$2l_{\kappa} K_{00 \gamma \delta} = (\xi_{\kappa} + \xi'_{\kappa}) K_{00 \gamma \delta}$$

from which follows (4.21).

Theorem 4.1. *If $K_{\alpha\gamma\delta}^{\beta}$ in $\bar{F}_n$ is $|\kappa$ recurrent, then also $\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}$ is $|\kappa$ recurrent with the same vector of recurrency $\bar{k}_{\kappa}$ i.e. (4.15) $\Rightarrow$*

$$(4.22) \quad (\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})|_{\kappa} = \bar{k}_{\kappa} (\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma}).$$

PROOF. From (4.15) and Lemma 4.2 follows

$$(K_{\alpha\beta\gamma\delta} + K_{\beta\alpha\gamma\delta})|_{\kappa} = (\bar{k}_{\kappa} + \mu_{\kappa})(K_{\alpha\beta\gamma\delta} + K_{\beta\alpha\gamma\delta}).$$

Substituting (2.13) into this equation we obtain

$$[(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})g_{\alpha\beta} + F\partial_{\epsilon} g_{\alpha\beta} K_{0\gamma\delta}^{\epsilon}]|_{\kappa} = (\bar{k}_{\kappa} + \mu_{\kappa})[(\lambda_{\gamma|\delta} - \lambda_{\delta|\gamma})g_{\alpha\beta} + F\partial_{\epsilon} g_{\alpha\beta} K_{0\gamma\delta}^{\epsilon}].$$

Contracting with l^{α} and l^{β} , and using (2.8d), we arrive to (4.22).

Lemma 4.5. *If $P_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent, then $P_{00\gamma\delta}$ is $|\kappa$ recurrent with the same vector of recurrency $\bar{p}_{\kappa}$.*

PROOF. In view of

$$(4.23) \quad P_{\alpha\gamma\delta}^{\beta}|_{\kappa} = \bar{p}_{\kappa} P_{\alpha\gamma\delta}^{\beta}$$

it follows from Lemma (4.1) that (4.2)–(4.6) remain valid if we write P and $\bar{p}$ in place of T and $\bar{i}$ respectively. (4.3) now has the form

$$P_{\alpha\beta\gamma\delta}|_{\kappa} = (\bar{p}_{\kappa} + \mu_{\kappa})P_{\alpha\beta\gamma\delta}.$$

Contracting the above equation first with l^{α} and then with l^{β} , and using (2.6d) we get

$$P_{00\gamma\delta}|_{\kappa} = (\bar{p}_{\kappa} - 2l_{\kappa})P_{00\gamma\delta} + (P_{0\kappa\gamma\delta} + P_{\kappa 0\gamma\delta})$$

Substituting the corresponding values from (2.17) and (2.20) into the above equation we obtain

$$(4.24) \quad P_{00\gamma\delta}|_{\kappa} = \bar{p}_{\kappa} P_{00\gamma\delta}.$$

Lemma 4.6. *If $P_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent with vector of recurrency $\bar{p}_{\kappa}$, and if exist vectorfields η_{κ} and η'_{κ} such that*

$$(4.25) \quad P_{\kappa\beta\gamma\delta} = \eta_{\kappa} P_{0\beta\gamma\delta},$$

$$(4.26) \quad P_{\beta\kappa\gamma\delta} = \eta'_{\kappa} P_{\beta 0\gamma\delta},$$

then $P_{0\beta\gamma\delta}$ and $P_{\beta 0\gamma\delta}$ are also $|\kappa$ recurrent:

$$(4.27) \quad P_{0\beta\gamma\delta}|_{\kappa} = (\bar{p}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa} + \eta_{\kappa})P_{0\beta\gamma\delta},$$

$$(4.28) \quad P_{\beta 0\gamma\delta}|_{\kappa} = (\bar{p}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa} + \eta'_{\kappa})P_{\beta 0\gamma\delta},$$

and

$$(4.29) \quad \eta_{\kappa} + \eta'_{\kappa} = 2l_{\kappa}.$$

PROOF. (4.27) and (4.28) are consequence of (4.25), (4.26) and Lemma 4.2. Contracting (4.25) and (4.26) with l^{β} , and substituting these equations into (2.17) we get

$$-(\eta_{\kappa} + \eta'_{\kappa})P_{00\gamma\delta} = l_{\kappa}(\lambda_{\epsilon}A_{\gamma\delta}^{\epsilon} + \lambda_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma}).$$

Substituting the value of (2.20) into the above equation we obtain (4.29).

Theorem 4.2. *If $P_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent, then also $\lambda_{\epsilon}A_{\gamma\delta}^{\epsilon} + \lambda_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma}$ is $|\kappa$ recurrent with the same vector of recurrency $\bar{p}_{\kappa}$.*

PROOF. From (4.23) and Lemma 4.1 we obtain

$$(P_{\alpha\beta\gamma\delta} + P_{\beta\alpha\gamma\delta})|_{\kappa} = (\bar{p}_{\kappa} + \mu_{\kappa})(P_{\alpha\beta\gamma\delta} + P_{\beta\alpha\gamma\delta}).$$

Contracting the above equation with l^{α} , and using (2.8d) we get

$$(P_{0\beta\gamma\delta} + P_{\beta 0\gamma\delta})|_{\kappa} - (P_{\kappa\beta\gamma\delta} - P_{\beta\kappa\gamma\delta}) = (\bar{p}_{\kappa} + 2^{-1}\mu_{\kappa} - l_{\kappa})(P_{0\beta\gamma\delta} + P_{\beta 0\gamma\delta}).$$

Contracting again with l^{β} , and using (2.8d), (2.17) and (2.20) we arrive to

$$(4.30) \quad (\lambda_{\epsilon}A_{\gamma\delta}^{\epsilon} + \lambda_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma})|_{\kappa} = \bar{p}_{\kappa}(\lambda_{\epsilon}A_{\gamma\delta}^{\epsilon} + \lambda_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma}),$$

i.e. (4.23) $\Rightarrow$ (4.30)

Lemma 4.7. *If $S_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent, then also $S_{00\gamma\delta}$ is $|\kappa$ recurrent with the same vector of recurrency $\bar{s}_{\kappa}$.*

PROOF. Since

$$(4.31) \quad S_{\alpha\gamma\delta}^{\beta}|_{\kappa} = \bar{s}_{\kappa}S_{\alpha\gamma\delta}^{\beta}$$

we have from Lemma 4.1

$$(4.32) \quad S_{00\gamma\delta}|_{\kappa} = (\bar{s}_{\kappa} - 2l_{\kappa})S_{00\gamma\delta} + (S_{\kappa 0\gamma\delta} + S_{0\kappa\gamma\delta}).$$

Substituting from (2.18) and (2.21) into this equation we obtain

$$(4.33) \quad S_{00\gamma\delta}|_{\kappa} = \bar{s}_{\kappa}S_{00\gamma\delta}.$$

Lemma 4.8. *If $S_{\alpha\gamma\delta}^{\beta}$ in $\bar{F}_n$ is $|\kappa$ recurrent with vector of recurrency $\bar{s}_{\kappa}$, and if there exist vectorfields ζ_{κ} and ζ'_{κ} such that*

$$(4.34) \quad S_{\kappa\beta\gamma\delta} = \zeta_{\kappa}S_{0\beta\gamma\delta},$$

$$(4.35) \quad S_{\beta\kappa\gamma\delta} = \zeta'_{\kappa}S_{\beta 0\gamma\delta},$$

then $S_{0\beta\gamma\delta}$, $S_{\beta 0\gamma\delta}$ are $|\kappa$ recurrent i.e.

$$(4.36) \quad S_{0\beta\gamma\delta}|_{\kappa} = (\bar{s}_{\kappa} + 2^{-1}\mu_{\kappa} - I_{\kappa} + \zeta_{\kappa})S_{0\beta\gamma\delta},$$

$$(4.37) \quad S_{\beta 0\gamma\delta}|_{\kappa} = (\bar{s}_{\kappa} + 2^{-1}\mu_{\kappa} - I_{\kappa} + \zeta'_{\kappa})S_{\beta 0\gamma\delta}$$

and

$$(4.38) \quad \zeta_{\kappa} + \zeta'_{\kappa} = 2I_{\kappa}.$$

PROOF. (4.36) and (4.37) are consequences of (4.34), (4.35) and Lemma 4.2 (formulae (4.10) and (4.11)). Contracting (4.34) and (4.35) with l^{β} , and substituting these equations into (2.18), we get

$$-(\zeta_{\kappa} + \zeta'_{\kappa})S_{00\gamma\delta} = (\mu_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma})I_{\kappa}.$$

Substituting (2.21) into this equation we obtain (4.38).

Theorem 4.3. If $S_{\alpha\gamma\delta}^{\beta}$ is $|\kappa$ recurrent, then also $\mu_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma}$ is $|\kappa$ recurrent with the same vector of recurrency $\bar{s}_{\kappa}$.

PROOF. Substituting (2.18) and (2.21) into (4.32) we obtain

$$(4.39) \quad (\mu_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma})|_{\kappa} = \bar{s}_{\kappa}(\mu_{\gamma}|_{\delta} - \mu_{\delta}|_{\gamma}),$$

i.e. (4.31) $\Rightarrow$ (4.39).

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