Cartan-type connections and connection sequences

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In a Finsler space (M, F) there exists a great variety of different metric connections. One of the most often used connection is Cartan connection. It has been characterized by certain conditions by M. MATSUMOTO [5], and also by B. HASSAN [3]. Many interesting aspect of these connections have been studied by J. GRIFONE [1, 2], I. Z. SZABÓ [8], R. MIRON [6], TAMÁSSY—KIS [9] and others. In the present paper by making use of a non linear connection N in the tangent bundle of M we slightly alser Hassan's third condition and we show that these conditions still uniquely determine a metric connection. The notion of a sequence of such connections also is introduced and its properties are investigated.

1. Notation and definition

Let M be an n dimensional differentiable manifold, τ_M be its tangent bundle and TM be the total space of τ_M. Let $\tilde{M}=TM-\{0\}$ be the manifold of all non-zero vectors on M, and $\pi\colon \tilde{M}\to M$ be the natural projection. We denote by $\pi^*(TM)$ the vector bundle induced naturally from TM by π (sometimes also denoted by $\pi^{-1}(TM)$). This is called the Finsler bundle over M, and its cross-sections are called Finsler vector fields on M.

Let $\mathscr{V}\tilde{M}=\{V\tilde{M}, \pi_V, \tilde{M}\}$ be the vertical bundle over $\tilde{M}$, where $V\tilde{M}$ is given by

$$V\tilde{M} = \ker(d\pi) = \bigcup_{z\in\tilde{M}} \ker(d\pi)_z.$$

Consider the vector bundle morphism

$$L\colon T\tilde{M} \to \pi^*(TM), \quad T_z\tilde{M}\ni X \to \big(z, (d\pi)(X)\big).$$

Suppose that a non-linear connection N, which is a Whitney-decomposition $T\tilde{M}:= N\oplus V\tilde{M}$ is given. In this case the resctriction $L\restriction N$ is a vector bundle isomorphism. We put $\beta:=(L\restriction N)^{-1}$ and call it the horizontal lift belonging to N.

For local calculation let (U, x^i) be a local coordinate system on M and let (x^i, y^i) be the induced local coordinates on $\pi^{-1}(U)$. Following HASSAN [3] we write $\partial_i(\tilde{m})$, $\underset{\cdot}{\partial}_i(\tilde{m})$ and $\bar{\partial}_i(\tilde{m})$ instead of $\left(\frac{\partial}{\partial x^i}\right)_{\tilde{m}}$, $\left(\frac{\partial}{\partial y^i}\right)$ and $\left(\tilde{m}, \left(\frac{\partial}{\partial x^i}\right)_{\pi(\tilde{m})}\right)$ respectively. Then $\{\partial_i(\tilde{m}), \underset{\cdot}{\partial}_i(\tilde{m})\}$ is a basis of $T_{\tilde{m}}\tilde{M}$ and $\{\bar{\partial}_i(\tilde{m})\}$ is a basis of the fibre

$\{(\tilde{m}, v) | v \in T_{(\pi)\tilde{m}} M\}$ of $\pi^*(TM)$. Evidently,

$$\partial_i, \dot{\partial}_i \colon \tilde{M} \to T\tilde{M}, \quad \tilde{m} \mapsto \partial_i(\tilde{m}) \quad \text{or} \quad \dot{\partial}_i(\tilde{m})$$

and

$$\bar{\partial}_i \colon \tilde{M} \to \pi^*(TM), \quad \tilde{m} \mapsto \bar{\partial}_i(\tilde{m})$$

are elements of $\mathscr{X}(\tilde{M})$ and $\operatorname{Sec} \pi^*(TM)$, resp. The Finsler vector field

$$\bar{v} \colon \tilde{M} \to \pi^*(TM), \quad \tilde{x} \mapsto (\tilde{x}, \tilde{x})$$

is called the fundamental field. In the above local coordinate system: $\bar{v} = y^i \bar{\partial}_i$.

2. Regular Finsler connections

A linear connection in the Finsler vector bundle $\pi^*(TM)$ is called a Finsler connection on the manifold M. Hence a Finsler connection is a map

$$\nabla \colon \mathscr{X}(\tilde{M}) \times \operatorname{Sec} \pi^*(TM) \to \operatorname{Sec} \pi^*(TM)$$

satisfying the following conditions:

(i) ∇ is R-bilinear.

(ii) For any C^∞-function $f\colon \tilde{M} \to \mathbf{R}$ and vector fields $\tilde{X} \in \mathscr{X}(M)$, $\bar{Y} \in \operatorname{Sec} \pi^*(TM)$ we have

$$\nabla_{f\tilde{X}} \bar{Y} = f \nabla_{\tilde{X}} \bar{Y} \quad \text{and} \quad \nabla_{\tilde{X}} f \bar{Y} = \tilde{X}(f) \bar{Y} + f \nabla_{\tilde{X}} \bar{Y}.$$

Also the notation $\nabla_{\tilde{X}} \bar{Y} := \nabla(\tilde{X}, \bar{Y})$ will be used.

Let ∇ be a Finsler connection. An element $\tilde{X}$ of $T\tilde{M}$ is called a *horizontal vector with respect to* ∇ iff $\nabla_{\tilde{X}} \bar{v} = 0$. We denote by H the set and bundle of all horizontal vectors, that is $H := \{\tilde{X} \in T\tilde{M} \,|\, \nabla_{\tilde{X}} \bar{v} = 0\}$.

A Finsler connection ∇ *is said to be regular* if $T\tilde{M}$ is the direct sum of H and $V\tilde{M}$. We say in this case that the non-linear connection H *is induced by* ∇.

Proposition 1. (Hassan [3]). *A Finsler connection is regular if and only if the map*

$$\nabla \bar{v} \colon V\tilde{M} \to \operatorname{Sec} \pi^*(TM), \quad V\tilde{M} \ni \tilde{X} \mapsto \nabla_{\tilde{X}} \bar{v}$$

is a linear isomorphism on the vertical vectors.

Let us denote by H the non-linear connection induced by a regular Finsler connection ∇ we have

Proposition 2. [3] *A Finsler connection* ∇ *is regular if and only if* $\det(\delta^i_j + y^k C^i_{jk}) \neq 0$. *In this case the connection parameters* H^i_j *of* H are *determined by*

$$H^i_j = y^k \Gamma^i_{jk},$$

where $\Gamma^i_{jk} = F^i_{jk} - H^l_j C^{il}_k$, *and* F^i_{jk}, C^i_{jk} *are the connection parameters of* ∇ *given by*

$$\nabla_{\partial_i} \bar{\partial}_j := F^k_{ij} \bar{\partial}_k \quad \text{and} \quad \nabla_{\dot{\partial}_i} \bar{\partial}_j := C^k_{ij} \bar{\partial}_k.$$

Proof. We have

$$\nabla_{\dot{\partial}_i} \bar{v} = \nabla_{\dot{\partial}_i} y^j \bar{\partial}_j = \delta^k_i \bar{\partial}_k + y^j C^k_{ij} \bar{\partial}_k = (\delta^k_i + y^j C^k_{ij}) \bar{\partial}_k$$

From Proposition 1 we get that ∇ is regular iff the mapping $\nabla\bar{v}$ maps the local basis $\{\dot{\partial}_i\}$ into a local basis. This is clearly equivalent to the given condition. — If the functions H_j^i are the connection parameters of H then the horizontal subbundle is generated by $\{\delta_i := \partial_i - H_i^j \dot{\partial}_j\}$ which implies that $\nabla_{\delta_i}\bar{v}=0$, from where

$$-H_i^j \delta_j^l \bar{\partial}_l + y^k \Gamma_{ik}^l \bar{\partial}_l = 0,$$
$$H_i^j = y^k \Gamma_{ik}^j.$$

3. Cartan-type connections (the main theorem)

Let

$$g\colon \operatorname{Sec}\pi^*(TM)\times\operatorname{Sec}\pi^*(TM) \to C^\infty(TM)$$
$$(\bar{X}, \bar{Y}) \mapsto g(\bar{X}, \bar{Y}) :\equiv \langle \bar{X}, \bar{Y}\rangle$$

be a symmetric (0, 2) Finsler tensor field. Suppose that g is non-degenerated. A Finsler connection ∇ is called compatible with g if $\nabla g=0$.

Theorem 1. *Let g be a non-degenerated symmetric* (0, 2) *Finsler tensor, and let N be an arbitrary non-linear connection on TM. There exists a unique Finsler connection ∇ in $\pi^*(TM)$ satisfying the following conditions:*

(I) $$\nabla g = 0 \quad \text{i.e.}\quad \forall \tilde{x}\in T\tilde{M},\quad \bar{Y}, \bar{Z}\in\operatorname{Sec}\pi^*(TM)$$
$$\tilde{X}\langle\bar{Y}, \bar{Z}\rangle - \langle\nabla_{\tilde{X}}\bar{Y}, \bar{Z}\rangle - \langle\nabla_{\tilde{X}}\bar{Z}, \bar{Y}\rangle = 0.$$

(II) If $\tilde{X}\in V\tilde{M}$, then for each $\tilde{Y}, \tilde{Z}\in T\tilde{M}$

$$\langle T(\tilde{X}, \tilde{Y}), L\tilde{Z}\rangle = \langle T(\tilde{X}, \tilde{Z}), L\tilde{Y}\rangle.$$

(III) For each $\tilde{X}, \tilde{Y}\in N$.

$$T(\tilde{X}, \tilde{Y}) = 0$$

where

$$T(\tilde{X}, \tilde{Y}) := \nabla_{\tilde{X}} L\tilde{Y} - \nabla_{\tilde{Y}} L\tilde{X} - L[\tilde{X}, \tilde{Y}],\quad \forall \tilde{X}, \tilde{Y}\in T\tilde{M}.$$

First we note that Hassan [3] has proved that for a given Finsler space (M, F) there exists a unique ∇ satisying the above conditions (I)—(III) with the alteration that $\tilde{X}, \tilde{Y}$ in (III) are elements of H, the horizontal bundle generated by ∇. Thus Theorem 1. is a generalization of Hassan's results in certain sense. (The connection determined by Hassan's theorem is just the Cartan connection.)

PROOF. If there exists a Finsler connection ∇ satisfying condition (I)—(III), then from (I) we have

$$\langle\nabla_{\tilde{X}}\bar{Y}, \bar{Z}\rangle = \tilde{X}\langle\bar{Y}, \bar{Z}\rangle - \langle\nabla_{\tilde{X}}\bar{Z}, \bar{Y}\rangle, \tag{1}$$
$$\forall \tilde{X}\in\mathscr{X}(\tilde{M}),\quad \bar{Y}, \bar{Z}\in\operatorname{Sec}\pi^*(TM).$$

Let $\tilde{X}^H$ and $\tilde{X}^V$ be the horizontal and vertical components of $\tilde{X}$ and thus $\tilde{X}=\tilde{X}^H+\tilde{X}^V$, then it follows from (1) that

$$\langle\nabla_{\tilde{X}}\bar{Y}, \bar{Z}\rangle = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle - \langle\nabla_{\tilde{X}^H}\bar{Z}, \bar{Y}\rangle + \tilde{X}^V\langle\bar{Y}, \bar{Z}\rangle - \langle\nabla_{\tilde{X}^V}\bar{Z}, \bar{Y}\rangle, \tag{2}$$

We consider $\beta\bar{Y}, \beta\bar{Z}$. It is clear that $\beta\bar{Y}, \beta\bar{Z}\in N$, and $L\circ\beta\bar{Y}=\bar{Y}$, $L\circ\beta\bar{Z}=\bar{Z}$. From (III) we obtain

$$\nabla_{\tilde{X}^H}\bar{Y}-\nabla_{\beta Z}L\tilde{X}^H-L[\tilde{X}^H, \beta\bar{Z}]=0.$$

These imply that

$$\langle\nabla_{\tilde{X}^H}\bar{Y}, \bar{Z}\rangle = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle-\langle\nabla_{\beta Z}L\tilde{X}^H, \bar{Y}\rangle-\langle L[\tilde{X}^H,\in\bar{Z}], \bar{Y}\rangle$$

$$(\text{from } I) = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle-\beta\bar{Z}\langle L\tilde{X}^H, \bar{Y}\rangle+\langle\nabla_{\beta Z}\bar{Y}, L\tilde{X}^H\rangle-\langle L[\tilde{X}^H, \beta\bar{Z}], \bar{Y}\rangle$$

$$(\text{from } II) = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle-\beta\bar{Z}\langle L\tilde{X}^H, \bar{Y}\rangle-\langle L[\tilde{X}^H, \beta\bar{Z}], \bar{Y}\rangle+$$

$$+\langle L[\beta\bar{Z}, \beta\bar{Y}], L\tilde{X}^H\rangle+\langle\nabla_{\beta Y}\bar{Z}, L\tilde{X}^H\rangle =$$

$$(\text{from } I) = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle-\beta\bar{Z}\langle L\tilde{X}^H, \bar{Y}\rangle-\langle L[\tilde{X}^H, \beta\bar{Z}], \bar{Y}\rangle+$$

$$+\langle L[\beta\bar{Z}, \beta\bar{Y}], L\tilde{X}^H\rangle+\beta\bar{Y}\langle L\tilde{X}^H, \bar{Z}\rangle-\langle\nabla_{\beta Y}L\tilde{X}^H, \bar{Z}\rangle$$

$$(\text{from } II) = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle+\beta\bar{Y}\langle L\tilde{X}^H, \bar{Z}\rangle-\beta\bar{Z}\langle L\tilde{X}^H, \bar{Y}\rangle+$$

$$+\langle L[\beta\bar{Z}, \beta\bar{Y}], L\tilde{X}^H\rangle-\langle L[\tilde{X}^H, \beta\bar{Z}], \bar{Y}\rangle-$$

$$-\langle L[\beta\bar{Y}, \tilde{X}^H], \bar{Z}\rangle-\langle\nabla_{\tilde{X}^H}\bar{Y}, \bar{Z}\rangle.$$

From this it follows

$$(*) \qquad 2\langle\nabla_{\tilde{X}^H}\bar{Y}, \bar{Z}\rangle = \tilde{X}^H\langle\bar{Y}, \bar{Z}\rangle+\beta\bar{Y}\langle\bar{Z}, L\tilde{X}^H\rangle-\beta\bar{Z}\langle L\tilde{X}^H, \bar{Y}\rangle+$$

$$+\langle L[\beta\bar{Z}, \beta\bar{Y}], L\tilde{X}^H\rangle-\langle L[\tilde{X}^H, \beta\bar{Z}], \bar{Y}\rangle-\langle L[\beta\bar{Y}, \tilde{X}^H], \bar{Z}\rangle.$$

On the other hand

$$T(\tilde{X}^V, \tilde{Y}) = \nabla_{\tilde{X}^V}L\tilde{Y}-\nabla_{\tilde{Y}}L\tilde{X}^V-L[\tilde{X}^V, \tilde{Y}] = \nabla_{\tilde{X}^V}L\tilde{Y}-L[\tilde{X}^V, \tilde{Y}].$$

and similarly

$$T(\tilde{X}^V, \tilde{Z}) = \nabla_{\tilde{X}^V}L\tilde{Z}-L[\tilde{X}^V, \tilde{Z}]$$

thus, in view of (II)

$$(3) \qquad \langle\nabla_{\tilde{X}^V}L\tilde{Y}-L[\tilde{X}^V, \tilde{Y}], L\tilde{Z}\rangle = (\nabla_{\tilde{X}^V}L\tilde{Z}-L[\tilde{X}^V, \tilde{Z}], L\tilde{Y}\rangle.$$

Now let $\tilde{Y}=\beta\bar{Y}$, $\tilde{Z}=\beta\bar{Z}$. From (3) we have

$$\langle\nabla_{\tilde{X}^V}\bar{Y}-L[\tilde{X}^V, \beta\bar{Y}], \bar{Z}\rangle = \langle\nabla_{\tilde{X}^V}\bar{Z}-L[\tilde{X}^V, \beta\bar{Z}], \bar{Y}\rangle,$$

which implies

$$(4) \qquad \langle\nabla_{\tilde{X}^V}\bar{Y}, \bar{Z}\rangle-\langle\nabla_{\tilde{X}^V}\bar{Z}, \bar{Y}\rangle = \langle L[\tilde{X}^V, \beta\bar{Y}], \bar{Z}\rangle-\langle L[\tilde{X}^V, \beta\bar{Z}], \bar{Y}\rangle$$

According to (I)

$$(5) \qquad \langle\nabla_{\tilde{X}^V}\bar{Y}, \bar{Z}\rangle+\langle\nabla_{\tilde{X}^V}\bar{Z}, \bar{Y}\rangle = \tilde{X}^V\langle\bar{Y}, \bar{Z}\rangle.$$

Adding the last two equations (4) and (5) we have

$$2\langle\nabla_{\tilde{X}^V}\bar{Y}, \bar{Z}\rangle = \tilde{X}^V\langle\bar{Y}, \bar{Z}\rangle+\langle L[\tilde{X}^V, \beta\bar{Y}], \bar{Z}\rangle-\langle L[\tilde{X}^V, \beta\bar{Z}], \bar{Y}\rangle$$

Finally, taking into account $(*)$ we obtain

$$2\langle\nabla_{\tilde X}\bar Y,\bar Z\rangle=2\langle\nabla_{\tilde X^H}\bar Y,\bar Z\rangle+2\langle\nabla_{\tilde X^V}\bar Y,\bar Z\rangle$$

$$(**)\qquad 2\langle\nabla_{\tilde X}\bar Y,\bar Z\rangle=\tilde X\langle\bar Y,\bar Z\rangle+\beta\bar Y\langle\bar Z,L\tilde X^H\rangle-$$
$$-\beta\bar Z\langle\bar Y,L\tilde X^H\rangle+\langle L[\beta\bar Z,\beta\bar Y],L\tilde X^H\rangle-\langle L[\tilde X^H,\beta\bar Z],\bar Y\rangle-$$
$$-\langle L[\beta\bar Y,\tilde X^H],\bar Z\rangle+\langle L[\tilde X^V,\beta\bar Y],\bar Z\rangle-\langle L\tilde X^V,\beta\bar Z],\bar Y\rangle.$$

Having the formula $(**)$, the assertions of the Theorem 1. can be concluded as follows.

a) Existence. Given $\tilde X\in\mathscr{X}(\tilde M)$ and $\bar Y\in\operatorname{Sec}\pi^*(TM)$ we define $\nabla_{\tilde X}\bar Y$ by $(**)$ which holds for every $\bar Z\in\operatorname{Sec}\pi^*(TM)$. It is straightforward to verity that the mapping $(\tilde X,\bar Y)\to\nabla_{\tilde X}\bar Y$ satisfies conditions (i), (ii) of paragraph 2. Hence this mapping determines a Finsler connection in $\pi^*(TM)$. By a calculation not detailed here and by the above definition of $\nabla_{\tilde X}\bar Y$, ∇ satisfies conditions (I)—(III).

b) Uniqueness. We have shown above that there exists a Finsler connection ∇ satisfying conditions (I)—(III), then it satisfies equation $(**)$. Hovwever, $(**)$ uniquely determines ∇ from N and g whose nondegeracy we have assumed. This completes the proof. □

4. Cartan-type connections (continuation)

Let ∇ be a Finsler connection, N be a non-linear connection on $\tilde M$, and g_{ij} be the components of a nondegenerated symmetric $(0, 2)$ Finsler tensor g. Setting

$$\nabla_{\delta_i}\bar\partial_j:=\Gamma^k_{ij}\bar\partial_k,\quad \nabla_{\dot\partial_i}\bar\partial_j:=C^k_{ij}\bar\partial_k,\quad \delta_i=\partial_i-N^j_i\partial_j$$

the following result follows from Theorem 1.

Corollary 1. *Let ∇ be a connection satisfying the conditions of Theorem 1, then the connection parameters of ∇ are determined by*

$$\Gamma^k_{ij}=\frac{1}{2}g^{pk}\{\delta_i(g_{jp})+\delta_j(g_{ip})-\delta_p(g_{ij})\}$$

and

$$C^k_{ij}=\frac{1}{2}g^{pk}\dot\partial_i(g_{jp}).$$

Using Proposition 2 we get the following statement.

Proposition 3. *The Finsler connection ∇ determined by Theorem 1 is regular if and only if*

$$\det\left(\delta^k_i+\frac{1}{2}y^j g^{fk}\frac{\partial g_{jk}}{\partial y^i}\right)\neq 0$$

Definition 1. A Finsler connection ∇ is said to be of *Cartan-type* if there exists a non-linear connection N for which ∇ is just the Finsler connection determined by N according to Theorem 1.

Theorem 2. Let $\mathring{\nabla}$ be a given Cartan-type connection, then the set of all Cartan-type connections is given by

$$N_i^k = \mathring{N}_j^k - A_i^k$$

$$C_{ij}^k = \mathring{C}_{ij}^k - \frac{1}{2} g^{pk} \frac{\partial g_{jk}}{\partial y^i},$$

$$\Gamma_{ij}^k = \mathring{\Gamma}_{ij}^k + g^{pk}(A_i^l C_{ljp} + A_j^l C_{lip} - A_p^l C_{lij}),$$

where $\mathring{N}_i^k, \mathring{C}_{ij}^k, \mathring{\Gamma}_{ij}^k$ (resp. $N_i^k, C_{ij}^k, \Gamma_{ij}^k$) are the connection-parameters of $\mathring{\nabla}$ (resp. ∇), A_i^k is an arbitrary Finsler tensor field, and $C_{ijk} := C_{ij}^l g_{lk}$.

Proof. Taking into account that $\delta_i = \mathring{\delta}_i + A_i^l \dot{\partial}_l$, from Corollary 1. one can easyly deduce the statement of Theorem 2.

A pair (M, g) is called a regular (generalized) Finsler space if

$$\det\left(\delta_i^k + \frac{1}{2} y^j \cdot g^{pk} \cdot \frac{\partial g_{jp}}{\partial y^i}\right) \neq 0.$$

The attributive "generalized" relates to the fact that g need not to be derived from a fundamental function F. It is clear that in the case of a regular generalized Finsler space every Cartan type connection is regular.

5. Connection sequences

In this § we consider a regular generalized Finsler space M. — Suppose that N is a non-linear connection on TM. According to Theorem 1, there exists a unique Cartan type connection $\overset{N}{\nabla}$ beloying to N. On the other hand, according to our above statement $\overset{N}{\nabla}$ is regular. Thus $\overset{N}{\nabla}$ induces a non-linear connection N_1. Applying again Theorem 1 we get $\overset{N_1}{\nabla}$ e.t.c. So we obtain the following connection-sequence

$$\text{(C.S)} \qquad N \to \overset{N}{\nabla} \to N_1 \to \overset{N_1}{\nabla} \to N_2 \to \dots$$

A connection-sequence $(C.\,S)$ is finite iff there exists an integer k such that $N_k = N_{k+1}$, or $\overset{N_k}{\nabla} = \overset{N_{k+1}}{\nabla}$.

Remark. The classical Cartan connection depends on g alone. It follows by Theorem 1 that if we start a connection-sequence $(C.S)$ with a non-linear connection N depending on g only, then every connection (non linear connection N_i and Finsler connections $\overset{N_i}{\nabla}$) belonging to $(C.S)$ depends on g alone, similarly to the classical Cartan connection.

Theorem 3. *If a connection-sequence $(C.\,S)$ contains only one non-linear conection (i.e. $N = N_1 = \dots$), then N_i^k must be the solution of the following system of equa-*

tions

(E.S.) $$N_i^k = \gamma_{ij}^k y^j + g^{pk}(N_p^l C_{ijl} - N_j^l C_{ilp} - N_i^l C_{ljp}) y^j$$

where

$$\gamma_{ij}^k = \frac{1}{2} g^{pk}\left(\frac{\partial g_{jp}}{\partial x^i} + \frac{\partial g_{ip}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right).$$

PROOF. According to Proposition 2. the coefficients of the non-linear connection N_1 induced by $\overset{N}{\nabla}$ are

$$N_{1i}^k = y^j \Gamma_{ij}^k,$$

where Γ_{ij}^k are the coefficients of $\overset{N}{\nabla}$. But by our assumption $N_1 = N$. Thus

$$N_i^k = y^j \Gamma_{ij}^k.$$

From the Corollary to Theorem 1. we immediately get

$$N_i^k = y^j \left\{\frac{1}{2} g^{pk}\big(\partial_i(g_{jp}) + \delta_j(g_{ip}) - \delta_p(g_{ij})\big)\right\},$$

or

$$N_i^k = \gamma_{ij}^k y^j + \frac{1}{2} y^j g^{pk}(N_p^l C_{ijl} - N_i^l C_{ljp} - N_j^l C_{ilp}). \quad \square$$

In case of change

$$x^{i'} = x^{i'}(x^1, \ldots, x^n)$$

$$y^{i'} = \frac{\partial x^{i'}}{\partial x^i} \cdot y^i$$

of the local coordinates on TM, by a local calculation we can prove the following.

Proposition 4. *If $\{N_i^k\}$ is a solution of (E.S) in local coordinates $\{x^i, y^i\}$, then*

$$\overline{N}_{k'}^{h'} := \frac{\partial x^{h'}}{\partial x^h} \cdot \frac{\partial x^k}{\partial x^{k'}} \cdot N_k^h + \frac{\partial x^{h'}}{\partial x^l} \cdot \frac{\partial^2 x^l}{\partial x^{k'} \cdot \partial x^{m'}} \cdot y^m$$

is a solution of (E.S) in the local coordinates $\{x^{i'}, y^{i'}\}$.

The last equation is nothing but the transformation low of a non-linear connection, and so we have the following result.

Corollary 3. If (E.S) has exactly one solution N_i^k, then N_i^k must be connection-parameters of a non-linear connection.

6. Classical Finsler space

Now we return to the case of classical Finsler spaces, when g originates from a fundamental fumction F, that is $g_{ij} = \frac{1}{2} \cdot \frac{\partial^2 F^2}{\partial y^i \cdot \partial y^j}$. Then we have

$$C_{ijk} y^i = C_{ijk} y^j = C_{ijk} y^k = 0$$

and (E.S) reduces to

$$N_i^k = \gamma_{ij}^k y^j - \frac{1}{2} \cdot g^{pk} N_j^l C_{lip} y^j \tag{6}$$

From this

$$N_i^k y^i = \gamma_{ij}^k y^i y^j := \gamma_{00}^k. \tag{7}$$

This implies that in (6) $N_j^l y^j = \gamma_{00}^l$, and so we have

$$N_i^k = \gamma_{ij}^k y^j - \frac{1}{2} g^{pk} \gamma_{00}^l C_{lip}. \tag{8}$$

It is easy to see that (8) is really a solution of (6). So (8) is the unique solution of (C.S), and according to Proposition 4. and its Corollary these N_i^k are connection parameters of a non-linear connection. Otherwise it is easy to check that the N_i^k given in (8) are nothing but the coefficients of the non-linear connection determined by the Cartan connection.

Now we start with a fundamental function F and determine the N_i^k as in (8), then according to Theorem 1. there exists a unique connection $\overset{N}{\nabla}$ satisfying conditions (I)—(III) of Theorem 1. This $\overset{N}{\nabla}$ generates a connection sequence (C.S), and thus $\overset{N}{\nabla} \to N_1$, where N_1 is the horizontal distribution of $\overset{N}{\nabla}$, i.e. $N_1 = H(\overset{N}{\nabla})$. But N is the solution of (E.S), then $N_1 = N$. Thus, in an other way, we arrive again to Hassan's theorem ([3], p. 17) mentioned also in this paper at the end of Theorem 1.

Finally we show that in an (M, F) a connection sequence can not be arbitrary long.

Theorem 4. *In the case of a classical Finsler space* (M, F) *every connection sequence* (C.S) *contains at most three different non-linear connections, that is in the connection sequence*

$$N \to \overset{N}{\nabla} \to N_1 \to \overset{N_1}{\nabla} \to N_2 \to \overset{N_2}{\nabla} \to N_3 \to \dots,$$

we have $N_2 = N_3 = \dots$. *Moreover* $\overset{N_2}{\nabla}$ *is just the Cartan connection.*

Proof. From Proposition 2. and the Corollary 1. in the case of a classical Finsler space (M, F) we have

$$N_{1j}^i = \Gamma_{jk}^i y^k = \gamma_{jk}^i y^k - \frac{1}{2} g^{pi} N_k^l C_{lpj} y^k \tag{9}$$

and in the analogy of this

$$N_{2j}^i = \Gamma_{1jk}^i y^k = \gamma_{jk}^i y^k - \frac{1}{2} g^{pi} N_{1k}^l C_{lpj} y^k. \tag{10}$$

From (9)

$$N_{1k}^l y^k = \gamma_{00}^l - \frac{1}{2} g^{pl} N_j^i C_{ipk} y^j y^k = \gamma_{00}^l$$

(because of $C_{ipk}y^k=0$) and thus from (10) we have

$$N^i_{2j} = \gamma^i_{jk}y^k - \frac{1}{2} g^{pi}\gamma^l_{00}C_{lpj}.$$

Thus N^i_{2j} is nothing but (8), hence $N_2=N_3=\dots$, and from Corollary 1. it follows that $\overset{N_2}{\nabla}$ is already the Cartan connection.

Acknowledgements: The present paper was prepared in the Matematical Institute of Debrecen University. I express my gratitude to Prof. L. TAMÁSSY and Dr. J. SZILASI for their help in preparing this paper.

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(Received February 28, 1986)