Characterization of additive functions with values in the circle group

By Z. DARÓCZY (Debrecen) and I. KÁTAI (Budapest)

1. Let G be a metrically compact Abelian group, and T the one dimensional torus. A function $\varphi\colon \mathbf{N}\to G$ will be called completely additive, if $\varphi(mn)=\varphi(m)+\varphi(n)$ holds for each couple of $m, n\in\mathbf{N}$. Let $\mathscr{A}_G^*$ be the class of completely additive functions.

Let F be a strictly monotonically increasing function defined on $\mathbf{N}$ and taking positive integer values for each large integer, i.e. $F(n)\in\mathbf{N}$ if $n>n_0$. Let $\{x_\nu\}_{\nu=1}^\infty$ be an infinite sequence in G. We shall say that it is of property $D[F]$, if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the subsequence $\{x_{\nu_{F(n)}}\}_{n=1}^\infty$ has also a limit. We say that it is of property $\Delta[F]$ if $\{x_{F(n)}-x_n\}_{n=1}^\infty$ is convergent. It is clear, that if $\{x_n\}\in\Delta[F]$, then $\{x_n\}\in D[F]$. For the linear function $F(n)=an+b$, $a\in\mathbf{N}$, $b\in Z$ we shall write $D[a, b]$, $\Delta[a, b]$ instead of $D[an+b]$, $\Delta[an+b]$.

Let $\mathscr{A}_G^*(D[F])$, $\mathscr{A}_G^*(\Delta[F])$ be the classes of those $\varphi\in\mathscr{A}_G^*$ for which $\{x_n=\varphi(n)\}_{n=1}^\infty$ is of property $D[F]$, $\Delta[F]$, respectively.

We are interested in giving a complete determination of $\mathscr{A}_G^*(D[F])$, $\mathscr{A}_G^*(\Delta(F))$.

We considered this problem with $F(n)=n+1$ in some earlier papers [1—4]. Recently E. WIRSING [5] proved that $\varphi\in\mathscr{A}_T^*(D[0, 1])$ if and only if

$$\varphi(n)\equiv\tau\log n \pmod 1 \quad (n\in\mathbf{N}) \tag{1.1}$$

for a $\tau\in\mathbf{R}$.

In [1] we proved that $\mathscr{A}_G^*(\Delta[0, 1])=\mathscr{A}_G^*(D[0, 1])$, and by using WIRSING's theorem in [2] we deduced the following assertion: If $\varphi\in\mathscr{A}_G^*(\Delta[0, 1])=\mathscr{A}_G^*(D[0, 1])$, then there exists a continuous homomorphism $\Psi\colon \mathbf{R}_x\to G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that φ is a restriction of Ψ on the set $\mathbf{N}$, i.e. $\varphi(n)=\Psi(n)$ $(\forall n\in\mathbf{N})$. The converse assertion is obvious. If $\Psi\colon \mathbf{R}_x\to G$ is a continuous homomorphism, then $\varphi(n):=\Psi(n)\in\mathscr{A}_G^*(\Delta[0, 1])\subseteq\mathscr{A}_G^*(D[0, 1])$. These results have been extended for additive functions (omitting the completeness of additivity) in [3], [4].

The case $F(n)=n+b$ can be treated similarly as $F(n)=n+1$. In this paper we shall investigate the case $F(n)=2n-1$. A complete determination of $\mathscr{A}_G^*(\Delta(2, -1])$ can be given easily, by using previous results [1], [2], [5] (see Section 4).

The characterization of $\mathscr{A}_G^*(D[2, -1])$ seems to be more complicated. We can solve it for $G=T$, (Theorem).

In the last section we shall formulate some conjectures.

2. Let $\varphi\in\mathscr{A}_G^*(D[2,-1])$. Let X denote the set of limit points of $\{\varphi(n)|n\in\mathbf{N}\}$, i.e. $g\in X$ if there exists $n_1<n_2<\ldots$, $n_\nu\in\mathbf{N}$, for which $\varphi(n_\nu)\to g$. Let $\varphi(2n_\nu-1)\to g'$. Then g' is determined by g. So the correspondence $L\colon g\to g'$ is a function. Let $X_0(\subseteq X)$ be the set of limit points of $\{\varphi(2m+1)|m\in\mathbf{N}\}$. Then $L\colon X\to X_0$, $L[X]=X_0$, and as it is easy to see, L is a continuous function. It is clear that X and X_0 are closed semigroups in G, so by a known theorem (see [5], Theorem (9.16)) they are compact groups.

For the proof of these simple assertions see [1].

Since $0\in X_0\subset X$, we have $\varphi(n)\in X$, $\varphi(2n+1)\in X_0$ for each $n\in\mathbf{N}$.

Lemma 1. *Let* $S\colon X\to X_0$ *be defined by*

$$S(g)=L\big(2g+\varphi(2)\big)-L(g). \tag{2.1}$$

If $n_1<n_2<\ldots$, $n_\nu\in\mathbf{N}$ *is such a sequence for which* $\varphi(n_\nu)\to g$, *then* $\varphi(2n_\nu+1)\to S(g)$ $(\nu\to\infty)$.

Proof. Since $\varphi\in\mathscr{A}_G^*(D[2,-1])$, $\varphi(n_\nu)\to g$ implies that $\varphi(2n_\nu^2)\to\varphi(2)+2g$,

$$\varphi\big(2\cdot(2n_\nu^2)-1\big)\to L\big(\varphi(2)+2g\big),$$

$\varphi(2n_\nu-1)\to L(g)$, $\varphi(2n_\nu+1)=\varphi\big(2\cdot(2n_\nu^2)-1\big)-\varphi(2n_\nu-1)\to S(g)$. □

Corollary 1. *We have*

$$S\big(-\varphi(2)\big)=0. \tag{2.2}$$

Proof. Substitute $g=-\varphi(2)$ in (2.1). □

Let $g\in X$, $n_1<n_2<\ldots$ be such a sequence for which $\varphi(n_\nu)\to g$. Let $m_\nu=2n_\nu$, $l_\nu^{(k)}=m_\nu^k+m_\nu^{k-1}+\ldots+m_\nu+1$. Then $\varphi(m_\nu)\to\varphi(2)+g$, $\varphi(l_\nu^{(1)})\to S(g)$. We have $l_\nu^{(k)}=1+m_\nu l_\nu^{(k-1)}=1+2n_\nu\cdot l_\nu^{(k-1)}$, and so the limits $\varphi(l_\nu^{(k)})\to\lambda_k$ $(\nu\to\infty)$ exist and

$$\lambda_k=S(g+\lambda_{k-1}),\quad k\geqq 1,\quad \lambda_0=0. \tag{2.3}$$

Since $\varphi(2^{k-1}n_\nu^k)\to(k-1)\varphi(2)+kg$, we have

$$\varphi(m_\nu^k-1)\to L\big((k-1)\varphi(2)+kg\big),\quad\text{and by}\quad m_\nu^k-1=(m_\nu-1)\,l_\nu^{(k-1)}$$

and we get

$$L\big((k-1)\varphi(2)+kg\big)=L(g)+\lambda_{k-1}\quad(k\geqq 1) \tag{2.4}$$

Lemma 2. *Let* $E_0=\{h|S(h)=0\}$. *Then* E_0 *is closed.*
Furthermore, $h\in E_0$ *if and only if*

$$L\big((k-1)\varphi(2)+kh\big)=L(h),\quad\forall k\in Z \tag{2.5}$$

$$L\big(-\varphi(2)\big)=L(h). \tag{2.6}$$

Proof. It is clear that E_0 is closed. If (2.5) holds for $k=2$, then $S(h)=0$, $h\in E_0$. Let us assume that $h\in E_0$. Then (2.5) holds with $k=2$. Apply now (2.3), (2.4) with h instead of g. We have $\lambda_k=0$ for each $k\geqq 1$, and (2.4) gives (2.5) for positive integers k. Let $U_h=\big\{k\big(\varphi(2)+h\big)\big|k\in\mathbf{N}\big\}$. So we proved that

$$L(h)=L\big(u-\varphi(2)\big)\quad\forall u\in U_h. \tag{2.6}$$

Let $\overline{U}_h$ be the smallest closed set that contains U_h. Since L is a continuous function and U_h is a semigroup, U_h is a closed semigroup, therefore by the cited theorem in [5], we have that $\overline{U}_h$ is a compact group. Since $\varphi(2)+h\in U_h$, the whole cyclic group $\{k(\varphi(2)+h)|k\in z\}\subseteq\overline{U}_h$, consequently (2.5) holds for negative k's as well. For $k=0$ we have (2.6). □

Let

$$\underline{R}=\{R_1<R_2<\dots\} \tag{2.7}$$

be an arbitrary infinite sequence of positive integers. We shall say that $\underline{R}$ belongs to $\mathscr{P}_0$ if for each $d\in\mathbf{N}$, d divides R_n for every large ν, i.e. if $\nu>\nu_0(\underline{R},d)$. Let $\tilde{\mathscr{P}}_0(\subseteq\mathscr{P}_0)$ be the set of those $\underline{R}\in\mathscr{P}_0$ for which $\lim\limits_n\varphi(R_n)$ exists. For an arbitrary sequence $\underline{R}$ let

$$a(\underline{R})=\lim_{\nu\to\infty}\varphi(R_\nu),$$

if the limit exists.

Let $a(\tilde{\mathscr{P}}_0)$ be the set of all limit points of a $(\underline{R})$, $\underline{R}\in\tilde{\mathscr{P}}_0$. If $\underline{R}\in\tilde{\mathscr{P}}_0$ and $d\in\mathbf{N}$, then $d\underline{R}=\{dR_1<dR_2<\dots\}\in\tilde{\mathscr{P}}_0$, and $a(d\underline{R})=\varphi(d)+a(\underline{R})$. If $\underline{R}\in\tilde{\mathscr{P}}_0$, $\underline{S}=\{S_1<S_2<\dots\}\in\mathscr{P}_0$, then $\underline{R}*\underline{S}=\{R_1S_1<R_2S_2<\dots\}\in\tilde{\mathscr{P}}_0$, $a(\underline{R}*S)=a(\underline{R})+a(\underline{S})$. So the set $\{a(\underline{R})|\underline{R}\in\tilde{\mathscr{P}}_0\}$ is a semigroup, the set of limit points is a semigroup as well, it is closed, so it is a compact group. Furthermore, from $a(d\underline{R})=\varphi(d)+a(\underline{R})$ we have that $\varphi(d)\in a(\tilde{\mathscr{P}}_0)$, consequently $X\subseteq a(\tilde{\mathscr{P}}_0)$. The relation $a(\tilde{\mathscr{P}}_0)\subseteq X$ is obvious. So we have $X=a(\tilde{\mathscr{P}}_0)$.

Let $\underline{M}\in\tilde{\mathscr{P}}_0$, $a(\underline{M})=g$. Let k be an arbitrary natural number. For each large ν we have $M_\nu=2kt_\nu$, $t_\nu\in\mathbf{N}$, $t_\nu<t_{\nu+1}$.

Now

$$\varphi(t_\nu)\to g-\varphi(2k)\quad(\nu\to\infty).$$

So we have

$$\varphi(M_\nu+k)=\varphi(2kt_\nu+k)=\varphi(k)+\varphi(2t_\nu+1)\to\varphi(k)+S(g-\varphi(2k)),$$

and so

$$\varphi(M_\nu+2m)\to\varphi(2m)+S(g-\varphi(4m)) \tag{2.8}$$

$$\varphi(M_\nu+2m+1)\to\varphi(2m+1)+S(g-\varphi(2m+1)) \tag{2.9}$$

$$\varphi(M_\nu+2m-1)\to\varphi(2m-1)+S(g-\varphi(2(2m-1))). \tag{2.10}$$

Now we observe the following relation. If $n_1<n_2<\dots$ is such a sequence of integers for which $\varphi(2n_\nu)\to\varkappa$, then $\varphi(2n_\nu+1)\to S(\varkappa-\varphi(2))$. This follows immediately from the relation $\varphi(n_\nu)\to\varkappa-\varphi(2)$.

Let us apply now this with $2n_\nu=M_\nu+2m$. From (2.8), (2.9) we get that

$$S(\varphi(m)+S(g-\varphi(4m)))=\varphi(2m+1)+S(g-\varphi(2(2m+1))). \tag{2.11}$$

Substitute $g=\varphi(2m)$ in (2.10) and observe (2.2):

$$S(\varphi(m))=\varphi(2m+1)+S(\varphi(m)-\varphi(2m+1)). \tag{2.12}$$

Since S is continuous, and (2.11) is true for every m, it is true for the limit points as well. Let $\tau\in X$, $\varphi(m_\nu)\to\tau$. Then $\varphi(2m_\nu+1)\to S(\tau)$, and from (2.12),

$$S(\tau)=S(\tau)+S(\tau-S(\tau))$$

which implies that

$$S(\tau - S(\tau)) = 0 \quad \forall \tau \in X. \tag{2.13}$$

If $\varphi(2n_\nu) \to \varkappa$, then $\varphi(n_\nu) \to \varkappa - \varphi(2)$, and so $\varphi(2n_\nu - 1) \to L(\varkappa - \varphi(2))$. Apply this with $2n_\nu = M_\nu + 2m$. From (2.9), (2.10) we get that

$$L(\varphi(m) + S(g - \varphi(4m))) = \varphi(2m-1) + S(g - \varphi(2(2m-1))).$$

Let $g = \varphi(2m)$. By (2.2) we have

$$L(\varphi(m)) = \varphi(2m-1) + S(\varphi(m) - \varphi(2m-1)),$$

and so by using the continuity of L and S, we get

$$S(\tau - L(\tau)) = 0 \quad \forall \tau \in X. \tag{2.14}$$

Let $g \in X$, $\varphi(n_\nu) \to g$. Then $\varphi(2n_\nu - 1) \to L(g)$, $\varphi(2n_\nu + 1) \to S(g)$,

$$S(L(g)) \leftarrow \varphi(2(2n_\nu - 1) + 1) = \varphi(4n_\nu - 1) \to L(\varphi(2) + g),$$
$$L(S(g)) \leftarrow \varphi(2(2n_\nu + 1) - 1) = \varphi(4n_\nu + 1) \to S(\varphi(2) + g).$$

So we have proved

Lemma 3. *We have*

$$SL(g) = L(\varphi(2) + g), \tag{2.15}$$

$$LS(g) = S(\varphi(2) + g), \tag{2.16}$$

$\forall g \in X$.

Lemma 4. *Assume that* $S(g) = L(g)$ $\forall g \in X_0$. *Then*

$$S(g) = L(g) \quad \forall g \in X. \tag{2.17}$$

Proof. Let $Y_k = k\varphi(2) + X_0$ $(k = 0, 1, 2, \ldots)$. We shall prove that (2.17) holds for $g \in Y_k$ $(k = 0, 1, \ldots)$.

We shall use the relations (2.15), (2.16). By the assumption this is true for $k = 0$. Let now $g \in Y_1$.

Then $g = \varphi(2) + h$ with a suitable $h \in Y_0$. So $S(h) = L(h)$.

Since $S(h) \in Y_0$, we have $L(S(h)) = S(L(h))$, i.e. our assertion is true for $k = 1$. Assume that (2.17) is proved for $g \in Y_0 \cup Y_1 \cup \ldots \cup Y_{k-1}$. Let $g \in Y_k$. Then $g = \varphi(2) + h$, $h \in Y_{k-1}$. Then $L(h) = S(h) \in Y_0$, and so $SL(h) = LS(h)$, and so (2.17) is true for each $g \in Y_k$. Since $\cup Y_k$ is everywhere dense in X, and S, G are continuous functions, therefore (2.17) is true for each $g \in X$. $\square$

Lemma 5. *Let* $h \in X$. *We have* $L(h) = 0$ *if and only if* $S(h) = 0$.

Proof. Assume that $L(h) = 0$. Then, by (2.14) we have $S(h) = 0$. It is clear that there exists at least one h^*, such that $L(h^*) = 0$. Then $h^* \in E_0$, and so by (2.6), $L(-\varphi(2)) = L(h^*) = 0$. So we have $L(-\varphi(2)) = 0$. Let now $S(h) = 0$, i.e. $h \in E_0$. Then by (2.6) we have $L(h) = 0$.

Lemma 6. *Let* V_g *denote the closure of the set* $\{kg | k \in \mathbf{N}\}$, $g \in X$. *Then* V_g *is a compact subgroup of* X. *If* $h \in E_0$, *then* $-\varphi(2) + V_{h+\varphi(2)} \subseteq E_0$.

PROOF. The assertion that V_g is a compact subgroup is well-known. If $h\in E_0$, then by Lemma 2 and Lemma 5 we have $(k-1)\varphi(2)+kh=k(h+\varphi(2))-\varphi(2)\in E_0$ for each $k\in Z$. Since L, S are continuous, our assertion follows immediately. □

Lemma 7. *If E_0 contains the only element $-\varphi(2)$, then $S(\tau)=L(\tau)=\tau+\varphi(2)$ $\tau\in X$, furthermore $\varphi(n+1)-\varphi(n)\to 0$ as $n\to\infty$.*

PROOF. Since $E_0=\{-\varphi(2)\}$, from (2.13), (2.14) we have $\tau-S(\tau)=-\varphi(2)$, $\tau-L(\tau)=-\varphi(2)$ $\forall\tau\in X$. Let now $\varkappa$, ϱ be such pairs of elements in X, for which there exists a sequence $n_1<n_2<\dots$ such that $\varphi(n_\nu)\to\varkappa$, $\varphi(n_\nu+1)\to\varrho$. Then $L(\varrho)\leftarrow\leftarrow\varphi(2n_\nu+1)\to S(\varkappa)$, and so $S(\varkappa)=L(\varrho)$, $S(\varkappa)=\varkappa+\varphi(2)$, $L(\varrho)=\varrho+\varphi(2)$, i.e. $\varkappa=\varrho$. Hence we can deduce easily that for each convergent sequence $\varphi(n_\nu)$ the sequence $\varphi(n_\nu+1)$ converges as well, $\lim\varphi(n_\nu)=\varkappa$ implies $\lim\varphi(n_\nu+1)=\varkappa$, and this gives almost immediately that $\varphi(n+1)-\varphi(n)\to 0$ $(n\to\infty)$. □

Let $Z=S[X_0]$, $W=L[X_0]$. It is clear that $g\in Z$, if there exists a suitable sequence $n_\nu\in\mathbf{N}$ such that $\varphi(2n_\nu-1)\to h$, and $S(h)=g\leftarrow\varphi(2(2n_\nu-1)+1)=\varphi(4n_\nu-1)$.

So Z=closure of $\{\varphi(4n-1)|n\in\mathbf{N}\}$, and similarly W=closure of $\{\varphi(4n+1)|n\in\mathbf{N}\}$.

Since the set $\{4n+1,\ n=0,1,2,\dots\}$ is a semigroup, $\{\varphi(4n+1)|n=0,1,2,\dots\}$ is a semigroup, and so W is a subgroup in X_0. It is clear that

$$\varphi(4n-1)+\varphi(4m+1)\in Z,$$

for each n, m, consequently

$$Z+W\subseteqq Z.$$

Similarly, $\varphi(4n-1)+\varphi(4m-1)\in W$, and so $Z+Z\subseteqq W$. Since $X_0=L[X_0]\cup S[X_0]$, $0\in X_0$, there exists $h\in X_0$ such that $L(h)=0$ or $S(h)=0$. From Lemma 5 we get that $0\in Z\cap W$. Then from the relations $Z+W\subseteqq Z$, $Z+Z\subseteqq W$ we have $Z=W=X_0$.

We have proved the following

Lemma 8. *We have*

$$L[X_0]=S[X_0]=X_0.$$

3. Let us consider now the special case $G=T$.

Assume that $\varphi\in\mathscr{A}_T^*(D[2,-1])$.

It is known that the only compact subgroups of T are T itself and the discrete groups $Z_m=\left\{\dfrac{k}{m}\ (\mathrm{mod}\ 1),\ k\in Z\right\}$.

Let us assume first that $X=T$. If there exists an $h\in E_0$ for which the order of the element $h+\varphi(2)$ is infinite, then $\{k(h+\varphi(2))|k\in\mathbf{N}\}$ is everywhere dense in T, and so by Lemma 6 we have $T\subseteqq E_0$, i.e. $L[T]=S[T]=0$, which leads to the trivial case $\varphi(n)=0$ $\forall n$, $(n,2)=1$.

Assume now that E_0 contains infinitely many elements h_j each of which has finite order $o(h_j)$. Then the orders $o(h_j)$ cannot be bounded, and so by Lemma 6 we have immediately that E_0 is everywhere dense in T. From the continuity of L and S we get that $T=E_0$, i.e. that $\varphi(n)=0$ $\forall n$, $(n,2)=1$.

There remains the case when $E_0=\{h_1,h_2,\dots,h_r\}$. If $r=1$, then Lemma 7 gives that $\varphi(n+1)-\varphi(n)\to 0$ $(n\to\infty)$, and Wirsing's theorem implies that $\varphi(n)\equiv\equiv\lambda\log n\ (\mathrm{mod}\ 1)$. Assume now that $r\geqq 2$. From $S(\tau-S(\tau))=0$ $\forall\tau\in T$, we get that $\tau - S(\tau) \in E_0$. Let B_j be the set of those τ, for which $\tau - S(\tau) = h_j$. The sets B_j are disjoint closed sets, $\cup B_j = T$, furthermore $\tau - S(\tau)$ is a continuous function on the whole T. This is impossible if $r \geqq 2$.

Let us assume now that $X = Z_M$ with a suitable M. If X_0 is the trivial group containing only the zero element, then $\varphi(2n+1) = 0$ for each $n \in \mathbf{N}$. Assume that X_0 contains at least two distinct elements. Since $L\colon X_0 \to X_0$, $S\colon X_0 \to X_0$ are such functions for which $L[X_0] = X_0$, $S[X_0] = X_0$, they are permutations in X_0. Consequently there exists a unique $\gamma_1 \in X_0$, and a unique $\gamma_2 \in X_0$ such that $L(\gamma_1) = 0$, $S(\gamma_2) = 0$. Since $L(\gamma_1) = 0$ implies that $S(\gamma_1) = 0$, we have $\gamma_1 = \gamma_2 =: \gamma$. If $\tau \in X_0$, then $\tau - L(\tau)$, $\tau - S(\tau) \in X_0$, and so from (2.13), (2.14) we get

$$S(\tau) = L(\tau) = \tau - \gamma \quad \forall \tau \in X_0. \tag{3.1}$$

Hence, by Lemma 4 we obtain

$$S(\tau) = L(\tau) \quad \forall \tau \in X. \tag{3.2}$$

Since X is a discrete group the conditions $\varphi \in \mathscr{A}_T^*(D[2, -1])$ and (3.2) imply that

$$\varphi(2n-1) = L(\varphi(n)) = S(\varphi(n)) = \varphi(2n+1)$$

for each large n. But from this we get immediately that $\varphi(n) = 0$ for each odd n.

Collecting our results we get the following

Theorem. *Let* $\varphi \in \mathscr{A}_T^*(D[-2, 1))$. *Then either*

(1) $\varphi(n) = 0$ *for each odd* n,

or

(2) $\varphi(n) \equiv \lambda \log n \pmod 1$ $(\forall n \in \mathbf{N})$ *with a suitable real number* λ.

Conversely, if $\varphi \in \mathscr{A}_T^*$ and satisfies (1) or (2), then $\varphi \in \mathscr{A}_T^*(D[-2, 1])$.

4. We are unable to prove a similar theorem for a general metrizable compact Abelian group.

The determination of $\mathscr{A}_G^*(\Delta[-2, 1])$ can be solved easily. Let $\varphi \in \mathscr{A}_G^*(\Delta[-2, 1])$. Then $\varphi(2n-1) - \varphi(n) \to c$, $\varphi(4n^2-1) - \varphi(2n^2) \to c$. Since $\varphi(4n^2-1) - \varphi(2n^2) = \varphi(2n-1) - \varphi(n) + (\varphi(2n+1) - \varphi(2n)) \to c$, we have $\varphi(2n+1) - \varphi(2n) \to 0$. Since $\varphi(2n+1) - \varphi(n+1) \to c$, the relations $\varphi(n+1) - \varphi(2n) \to -c$, $\varphi(n+1) - \varphi(n) \to \varphi(2) - c =: A$ also hold. Then

$$\varphi(n+1) - \varphi(n) = (\varphi(2n+2) - \varphi(2n+1)) + (\varphi(2n+1) - \varphi(2n)),$$

whence we get $A = 2A$, i.e. $A = 0$. So we have $\varphi \in \mathscr{A}_G^*(\Delta[0, 1])$.

5. Now we formulate some conjectures.

Conjecture 1. Let $\varphi \in \mathscr{A}_T^*$ and assume that the set $\{\varphi(n) | n \in \mathbf{N}\}$ is everywhere dense in T. Let $\eta_n := (\varphi(n), \varphi(n+1), \dots, \varphi(n+k))$, and assume that the sequence $\{\eta_n | n \in \mathbf{N}\}$ is not everywhere dense in $T_{k+1} = T \times \dots \times T$. Then $\varphi(n) = \lambda \log n \pmod 1$ with a suitable $\lambda \in \mathbf{R}$.

Conjecture 2. Let $\varphi, \Psi \in \mathcal{A}_T^*$, and assume that both of the sets $\{\varphi(n)|n\in\mathbf{N}\}$, $\{\Psi(n)|n\in\mathbf{N}\}$ are everywhere dense in T. Assume furthermore that the sequence $\xi_n=(\varphi(n), \Psi(n+1))$ is not everywhere dense in $T_2=T\times T$. Then there exist a suitable $\lambda\in\mathbf{R}$ and a rational s such that $\Psi(n)=s\varphi(n)$, $\varphi(n)\equiv\lambda \log n$ $\forall n\in\mathbf{N}$.

Conjecture 3. Let G_1, G_2 be metrically compact Abelian groups, $\varphi\in\mathcal{A}_{G_1}^*$, $\Psi\in\mathcal{A}_{G_2}^*$. Assume that the sets $\{\varphi(n)|n\in\mathbf{N}\}$, $\{\Psi(n)|n\in\mathbf{N}\}$ are everywhere dense in G_1, G_2, respectively. Assume furthermore that the sequence $\theta_n=(\varphi(n), \Psi(n+1))$ is not everywhere dense in $H=G_1\times G_2$. Then there exist integers P and Q such that

$$PG_1 = QG_2, \quad P\varphi(n) = Q\Psi(n),$$

and there exists a continuous homomorphism $\Lambda: R_x\to PG_1$ such that $\Lambda(n)= =P\varphi(n)=Q\Psi(n)$.

References

[1] Z. Daróczy and I. Kátai, On additive number-theoretical functions with values in a compact Abelian group, *Aequationes Mathematicae* **28** (1985), 288—292.

[2] Z. Daróczy and I. Kátai, On additive arithmetical functions with values in topological groups, *Publ. Math. Debrecen* **33** (1986), 287—292.

[3] Z. Daróczy and I. Kátai, On additive arithmetical functions with values in topological groups, II. *Publ. Math. Debrecen* **34** (1984), 65—68.

[4] Z. Daróczy and I. Kátai, On additive functions taking values from a compact group.

[5] E. Wirsing, The proof is given in a letter to I. Kátai (9.3. 1984).

[6] E. Hewitt—K. A. Ross, Abstract harmonic analysis, *Berlin* 1963, Springer.

Z. DARÓCZY
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF L. KOSSUTH
DEBRECEN 4010, HUNGARY

I. KÁTAI
EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BUDAPEST, H—1117
BOGDÁNFY U. 10/B

(Received February 13, 1987)