

Orthogonally additive mappings on free inner product Z -modules

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Abstract. It is known that, if X is a real inner product space of dimension at least 2 and Y an abelian group, every solution of the conditional Cauchy functional equation $(*)$ (see below) is additive if it is odd and is quadratic if it is even. In this paper the solutions of $(*)$ are determined if X is a special free inner product Z -module. If $\dim_Z X = 2$, Theorem 15 expresses a serious deviation from the situation in the inner product space case while Theorems 13 and 17 show that for $\dim_Z X$ sufficiently large, we have analogies to that case.

1. Introduction

If the sets X and Y are furnished with a binary operation $+$ and X furthermore with a binary relation $\perp$, called orthogonality, then a mapping $f : X \rightarrow Y$ is said to be *orthogonally additive* if it satisfies the conditional Cauchy functional equation

$$(*) \quad f(x + z) = f(x) + f(z) \quad \text{for all } x, y \in X \text{ with } x \perp z.$$

In recent applications, conditional functional equations generally play an increasingly important role.

Orthogonal additivity has one of its roots in inner product spaces, when $\perp$ stems from an inner product; for a brief survey, we refer to the

second half of the paper [7], where also the orthogonal additivity in the Blaschke–Birkhoff–James sense over normed spaces is mentioned. Several papers, the first being [8], treated orthogonal additivity under regularity conditions. By a complete change of the methods of proof, all the regularity conditions could be avoided a priori, and the following theorem was obtained:

Theorem 1. *If $(X, \langle \cdot, \cdot \rangle)$ is a real inner product space with $\dim_R X \geq 2$, if $x \perp z$ is defined by $\langle x, z \rangle = 0$ ($x, z \in X$), and if Y is an abelian group, then $f : X \rightarrow Y$ is a solution of $(*)$ if and only if there exist additive mappings $l : R \rightarrow Y$ and $h : X \rightarrow Y$ such that $f(x) = l(\|x\|^2) + h(x)$ ($\forall x \in X$) ([10], p. 43, Corollary 10; [14], Theorem 1; for more general versions [11], p. 242, 246).*

This result follows from theorems which separately treat the even and the odd case in the framework of an axiomatic theory of orthogonality spaces ([10], p. 38/39, Theorem 5 and 6; p. 41, Corollary 7). Roughly speaking, here and in important other situations, the general even solution of $(*)$ is quadratic and its general odd solution is additive. PINSKER's [8] and other regularity results then follow as corollaries from Theorem 1 ([10], p. 43–46). The following statement is also a consequence of Theorem 1; it extends the already long list of characterizations of Hilbert spaces among inner product spaces:

Corollary 2. *For a real inner product space $(X, \langle \cdot, \cdot \rangle)$ with $\dim_R X \geq 2$, the following are equivalent:*

- (i) *Every orthogonally additive $f : X \rightarrow R$ which is bounded below attains a minimum.*
- (ii) *X is a Hilbert space. ([10], p. 46, Corollary 15).*

Further applications of Theorem 1 are the Boltzmann–Gronwall Theorem in gas dynamics ([1], p. 191–194) and a premium calculation principle in actuarial science ([5], section 3).

One of the actual objectives in the theory of the functional equation $(*)$ is its investigation beyond the general theory developed in [10], [11], [12], [13] on vector spaces. It is the purpose of this paper to present results about orthogonal additivity on a special class of free Z -modules (cf. [6] for a related problem) and to compare them with those of the vector space case.

2. Notation and preliminaries

Throughout the paper, N , N^0 , Z , R denote the sets of positive integers, nonnegative integers, integers, real numbers, respectively. We use 0 for the identity element of the groups $(X, +)$ and $(Y, +)$ as well as for the integer zero; it will always be clear from the context what is meant. $\underline{c}$ is the symbol for the constant mapping with value c , and $:=$ means that the right hand side defines the left hand side. Finally, $=_{(\dots)}=$ is used for quoting the earlier result ($\dots$).

Remark 3. A free Z -module X is, up to isomorphism, a direct sum $Z^{(J)} := \bigoplus_{j \in J} X_j$ where $X_j = Z$ for all $j \in J$ and where J is an appropriate index set. The elements $e_j := (\delta_{jk})_{k \in J}$ ($j \in J$) (Kronecker symbols) constitute a basis of the free Z -module $Z^{(J)}$, the so-called canonical basis. Since Z is a commutative ring $\neq \{0\}$, every free Z -module X has a well-defined dimension $\dim_Z X$. Of course, $\dim_Z Z^{(J)} = \text{card } J$. ([3], p. 41, 42, 150, 151).

In the notation $(b_j)_{j \in J}$ for a basis of a free Z -module, we always assume $b_j \neq b_k$ for $j, k \in J$, $j \neq k$.

Definition 4. If $(b_j)_{j \in J}$ is a basis of a free Z -module X , then $\langle \cdot, \cdot \rangle : X \times X \rightarrow Z$ defined by $\langle x, z \rangle := \sum_{j \in J} p_j q_j$ ($x = \sum_{j \in J} p_j b_j \in X$, $z = \sum_{j \in J} q_j b_j \in X$) is called *the standard inner product on X associated with $(b_j)_{j \in J}$* . (Notice that all sums over J with running index j automatically contain only a finite number of nonzero summands). We briefly write $\sum$ for $\sum_{j \in J}$. $(X, (b_j)_{j \in J}, \langle \cdot, \cdot \rangle)$, sometimes more briefly denoted by X , is then said to be a *free standard inner product Z -module* (FSIP Z -module).

(0) For $x, z \in X$, we define $x \perp z : \Longleftrightarrow \langle x, z \rangle = 0$.

Lemma 5. *If $(X, (b_j)_{j \in J}, \langle \cdot, \cdot \rangle)$ is a FSIP Z -module, then we have:*

- a) $\langle \cdot, \cdot \rangle$ is Z -bilinear, symmetric, and positive definite.
- b) $\langle b_j, b_k \rangle = \delta_{jk}$ ($\forall j, k \in J$), i.e., $(b_j)_{j \in J}$ is an orthonormal basis.
- c) $x, z \in X$; $x \perp z$; $p, q \in Z \implies px \perp qz$.
- d) $j, k \in J$; $p, q \in Z \implies \langle p(b_j + b_k), q(b_j - b_k) \rangle = 0$.

The routine proof is omitted.

Definition 6. a) If X is a FSIP Z -module and $(Y, +)$ an abelian group, a mapping $f : X \rightarrow Y$ is called orthogonally additive if

$$(*) \quad f(x+z) = f(x) + f(z) \quad \text{for all } x, z \in X \text{ with } x \perp z$$

holds.

$\text{Hom}_\perp(X, Y)$ denotes the set of all solutions f of $(*)$,

(e) $\text{Hom}_\perp(X, Y) := \{g \in \text{Hom}_\perp(X, Y); g \text{ even}\}$,

(o) $\text{Hom}_\perp(X, Y) := \{h \in \text{Hom}_\perp(X, Y); h \text{ odd}\}$.

b) $\text{Hom}(X, Y) := \text{Hom}_Z(X, Y) = \{f : X \rightarrow Y; f(x+z) = f(x) + f(z)(x, z \in X)\}$ is the set of all *additive mappings* $f : X \rightarrow Y$.

c) $\text{Quad}(X, Y) := \{f : X \rightarrow Y; f \text{ satisfies } (JvN)\}$ where

$(JvN): f(x+z) + f(x-z) = 2f(x) + 2f(z)(\forall x, z \in X)$ is the set of all *quadratic mappings* $f : X \rightarrow Y$.

Definition 7. We say that the abelian group $(Y, +)$ is *uniquely 2-divisible* if the mapping $\omega : Y \rightarrow Y$, $\omega(y) := 2y (\forall y \in Y)$ is bijective. Then both ω and ω^{-1} are automorphisms of $(Y, +)$, and we write $\frac{1}{2}y$ for $\omega^{-1}(y)$.

Lemma 8. For a FSIP Z -module X and an abelian group $(Y, +)$ we have:

a) $\text{Hom}(X, Y) \subset (\text{o})\text{Hom}_\perp(X, Y) \subset \text{Hom}_\perp(X, Y) \subset \{f : X \rightarrow Y; f(0) = 0\}$.

b) $f \in \text{Hom}_\perp(X, Y); \tilde{f}(x) := f(-x)(\forall x \in X) \implies \tilde{f} \in \text{Hom}_\perp(X, Y)$.

c) $f, g \in \text{Hom}_\perp(X, Y) \implies f+g, f-g \in \text{Hom}_\perp(X, Y)$.

d) $(Y, +)$ uniquely 2-divisible, $f \in \text{Hom}_\perp(X, Y)$, $g(x) := \frac{1}{2}[f(x)+f(-x)]$, $h(x) := \frac{1}{2}[f(x)-f(-x)](\forall x \in X) \implies g \in (\text{e})\text{Hom}_\perp(X, Y)$, $h \in (\text{o})\text{Hom}_\perp(X, Y)$, $f = g + h$.

PROOF. a) The first two inclusions are evident. By (0), $0 \perp 0$, so $f(0) = f(0+0) = f(0) + f(0)$, i.e., $f(0) = 0$. — b) If $x, z \in X$, $x \perp z$, then by (0) $(-x) \perp (-z)$, and then $\tilde{f}(x+z) = f(-x-z) = f(-x) + f(-z) = \tilde{f}(x) + \tilde{f}(z)$. — c) is straightforward, and d) follows from b) and c).

3. Main results

Throughout this section we suppose that $(X, (b_j)_{j \in J}, \langle \cdot, \cdot \rangle)$ be a FSIP Z -module and $(Y, +)$ an abelian group.

Remark 9. We first separate the case $\dim_Z X \leq 1$ from the rest of the theory.

- a) If $\dim_Z X = 0$, then by Lemma 8a) $\text{Hom}_\perp(X, Y) = \{0\}$.
- b) If $\dim_Z X = 1$, $\{b\}$ a basis of X , then we have for $x, z \in X$, $x = pb$, $z = qb$ by Definition 4 $\langle x, z \rangle = pq$, therefore by (0)

$$(1) \quad x, z \in X \implies [x \perp z \iff x = 0 \text{ and/or } z = 0].$$

Let be $f : X \rightarrow Y$, $f(0) = 0$, and $x, z \in X$, $x \perp z$. By (1) $x = 0$ and/or $z = 0$, say $z = 0$. $f(x + z) = f(x) = f(x) + f(0) = f(x) + f(z)$. So $f \in \text{Hom}_\perp(X, Y)$, and together with Lemma 8a) we get

$$(2) \quad \text{Hom}_\perp(X, Y) = \{f : X \rightarrow Y; f(0) = 0\}.$$

Hence in the case $\dim_Z X \leq 1$, the determination of $\text{Hom}_\perp(X, Y)$ is completely settled.

Example 10. Let be $\dim_Z X = 1$, $\{b\}$ a basis of X , $Y \neq \{0\}$, $a \in Y \setminus \{0\}$. Define $h : X \rightarrow Y$ by

$$h(pb) = \begin{cases} a & (p > 0) \\ 0 & (p = 0) \\ -a & (p < 0) \end{cases} \quad (p \in Z).$$

Then $h(2b) = a$, $h(b) = a$, $2h(b) = 2a$, so $h(2b) \neq 2h(b)$. This shows that $h \notin \text{Hom}(X, Y)$. But by (2), $h \in \text{Hom}_\perp(X, Y)$ and h is odd, i.e.,

$$(o) \text{Hom}_\perp(X, Y) \not\subset \text{Hom}(X, Y),$$

which means that the first inclusion in Lemma 8a) may be strict.

Example 11. Let be $\dim_Z X = 1$, $\{b\}$ a basis of X , and assume that there exists $a \in Y$ such that $a \neq 4a$. Define $g : X \rightarrow Y$ by

$$g(pb) = \begin{cases} a & (p \in Z \setminus \{0\}) \\ 0 & (p = 0) \end{cases}.$$

Then $g(2b) = a$, $g(b) = a$, $4g(b) = 4a$, so $g(2b) \neq 4g(b)$. Assume that $g \in \text{Quad}(X, Y)$. Then put $x = z = b$ in (JvN) to obtain $g(2b) + g(0) = 2g(b) + 2g(b)$, i.e., $g(2b) = 4g(b)$, contradiction. So $g \notin \text{Quad}(X, Y)$, but by (2) $g \in \text{Hom}_\perp(X, Y)$, and g is even, therefore

$$(e) \text{Hom}_\perp(X, Y) \not\subset \text{Quad}(X, Y).$$

We now turn to the case $\dim_Z X \geq 2$ where we shall find situations contrasting with those in Examples 10 and 11.

Lemma 12. *If $f, g \in \text{Hom}_\perp(X, Y)$, then we have:*

$$(3) \quad f(x) = \sum f(p_j b_j) \quad \text{for } x = \sum p_j b_j \in X.$$

$$(4) \quad \begin{aligned} &g \text{ even; } j, k \in J, j \neq k; \\ &p \in Z \text{ even} \implies g(p b_j) = 2g\left(\frac{p}{2} b_j\right) + 2g\left(\frac{p}{2} b_k\right). \end{aligned}$$

PROOF. (3): By Lemma 8a), $f(0) = 0$, so a zero summand $p_j b_j$ of $\sum p_j b_j$ produces a zero summand of $\sum f(p_j b_j)$. (3) is a matter of finite sums and is established by induction on the number of nonzero summands starting from Lemma 5a), b), c) and from (*). — (4): $g(p b_j) = g[\frac{p}{2}(b_j + b_k) + \frac{p}{2}(b_j - b_k)] =_{L.5d} g[\frac{p}{2}(b_j + b_k)] + g[\frac{p}{2}(b_j - b_k)] =_{(3)} g(\frac{p}{2} b_j) + g(\frac{p}{2} b_k) + g(\frac{p}{2} b_j) + g(\frac{p}{2} b_k) = 2g(\frac{p}{2} b_j) + 2g(\frac{p}{2} b_k)$.

Theorem 13. *If $\dim_Z X \geq 2$, then $(o) \text{Hom}_\perp(X, Y) = \text{Hom}(X, Y)$.*

PROOF. i) $\text{Hom}(X, Y) \subset (o) \text{Hom}_\perp(X, Y)$ follows from Lemma 8a).
ii) Assume that $h \in (o) \text{Hom}_\perp(X, Y)$. By partially using a method in [4], p. 4.74/4.75, we show that

$$(5) \quad (H, j, p) : h(p b_j) = p h(b_j) \quad \text{holds for all } j \in J \text{ and all } p \in Z.$$

$p = 0$: By Lemma 8a) $h(0) = 0$, so $(H, j, 0)$ holds for all $j \in J$.

$p = 1$: $(H, j, 1)$ trivially holds for all $j \in J$.

Let be $n \in N$, $n \geq 2$, and assume that (H, j, p) holds for all $p \in N^0$ such that $p \leq n - 1$ and for all $j \in J$. Let be $j \in J$ arbitrary and choose

$k \in J \setminus \{j\}$ arbitrary (notice that $\dim_Z X \geq 2$). Then we get

$$\begin{aligned}
 (6) \quad & h[nb_j + (n-2)b_k] = h[(n-1)(b_j + b_k) + (b_j - b_k)] \\
 & \stackrel{\text{L.5d}}{=} h[(n-1)(b_j + b_k)] + h(b_j - b_k) \stackrel{(3)}{=} \\
 & = h[(n-1)b_j] + h[(n-1)b_k] + h(b_j) - h(b_k) \\
 & \stackrel{=(H,j,n-1),(H,k,n-1)}{=} (n-1)h(b_j) + (n-1)h(b_k) + h(b_j) - h(b_k) \\
 & = nh(b_j) + (n-2)h(b_k).
 \end{aligned}$$

On the other hand

$$\begin{aligned}
 (7) \quad & h[nb_j + (n-2)b_k] \stackrel{(3)}{=} h(nb_j) + h[(n-2)b_k] \stackrel{=(H,k,n-2)}{=} \\
 & = h(nb_j) + (n-2)h(b_k).
 \end{aligned}$$

From (6) and (7) we obtain $h(nb_j) = nh(b_j)$. Analogously, with j, k interchanged, $h(nb_k) = nh(b_k)$, so (H, j, n) holds for all $n \in N^0$ and all $j \in J$, i.e.,

$$(8) \quad h(nb_j) = nh(b_j) (\forall n \in N^0, \forall j \in J).$$

Let be $p \in Z$, $p < 0$, and $j \in J$ arbitrary. Then $h(pb_j) = h(-(-p)b_j) = -h((-p)b_j) \stackrel{(8)}{=} ph(b_j)$. This and (8) imply (5).

Let be $x, z \in X$ arbitrary, say $x = \sum p_j b_j$, $z = \sum q_j b_j$, so $x + z = \sum (p_j + q_j) b_j$, and we have $h(x + z) \stackrel{(3)}{=} \sum h[(p_j + q_j)b_j] \stackrel{(5)}{=} \sum (p_j + q_j)h(b_j) = \sum p_j h(b_j) + \sum q_j h(b_j) \stackrel{(5)}{=} \sum h(p_j b_j) + \sum h(q_j b_j) \stackrel{(3)}{=} h(\sum p_j b_j) + h(\sum q_j b_j) = h(x) + h(z)$. Therefore $h \in \text{Hom}(X, Y)$, and (o) $\text{Hom}_\perp(X, Y) \subset \text{Hom}(X, Y)$ is established, which completes the proof.

Remark 14. A more explicit form of $h \in \text{Hom}(X, Y)$ is of course

$$(9) \quad h(x) = \sum p_j c_j \text{ for } x = \sum p_j b_j \in X \text{ and } c_j := h(b_j) \quad (\forall j \in J).$$

Theorem 15. If $\dim_Z X = 2$ and the basis $(b_j)_{j \in J}$ is denoted by (b_1, b_2) , then $g : X \rightarrow Y$ belongs to $(e) \text{Hom}_\perp(X, Y)$ if and only if there

exist $a_1, a_2 \in Y$ such that

$$(10) \quad g(p_1 b_1 + p_2 b_2) = g(p_1 b_1) + g(p_2 b_2) \quad (\forall p_1, p_2 \in Z),$$

$$(11) \quad g(p b_1) = g(p b_2) = \frac{1}{2} p^2 (a_1 + a_2) \quad (p \in Z \text{ even}),$$

$$(12) \quad g(p b_1) = \frac{1}{2} (p^2 + 1) a_1 + \frac{1}{2} (p^2 - 1) a_2 \quad (p \in Z \text{ odd}),$$

$$(13) \quad g(p b_2) = \frac{1}{2} (p^2 - 1) a_1 + \frac{1}{2} (p^2 + 1) a_2 \quad (p \in Z \text{ odd}).$$

PROOF. i) Assume that $g \in (\text{e}) \text{Hom}_\perp(X, Y)$. (10) is implied by (3), and (4) leads to

$$(14) \quad g(p b_1) = g(p b_2) = 2g\left(\frac{p}{2} b_1\right) + 2g\left(\frac{p}{2} b_2\right) \quad (p \in Z \text{ even}).$$

We put

$$(15) \quad a_1 := g(b_1), \quad a_2 := g(b_2)$$

and prove (11), (12), (13) by induction on p ; evenness of g implies that we need only consider $p \in N^0$. $g(0) = 0$ (from Lemma 8a)) and (15) guarantee that (11), (12), (13) hold for $p = 0$, $p = 1$, respectively.

Let be $p \in N$ odd and assume that (11), (12), (13) hold for $g(r b_1)$, $g(s b_2)$ where $0 \leq r, s \leq p$. Since $p + 1$ is even, (14) yields

$$(16) \quad g[(p+1)b_1] = g[(p+1)b_2] = 2g\left(\frac{p+1}{2} b_1\right) + 2g\left(\frac{p+1}{2} b_2\right).$$

From $1 \leq p$ we conclude $\frac{p+1}{2} \leq p$ so that (11), (12), (13) can be applied to the right hand side of (16).

Case 1: $\frac{p+1}{2}$ is even. By (11) for $\frac{p+1}{2}$ we get

$$(17) \quad g\left(\frac{p+1}{2} b_1\right) = g\left(\frac{p+1}{2} b_2\right) = \frac{1}{2} \left(\frac{p+1}{2}\right)^2 (a_1 + a_2),$$

so $g[(p+1)b_1] = g[(p+1)b_2] \stackrel{(16), (17)}{=} 2 \left(\frac{p+1}{2}\right)^2 (a_1 + a_2) = \frac{1}{2} (p+1)^2 (a_1 + a_2)$, which means that $(11)_{p+1}$ holds here.

Case 2: $\frac{p+1}{2}$ is odd. By (12), (13) for $\frac{p+1}{2}$ we get

$$(18) \quad g\left(\frac{p+1}{2}b_1\right) = \frac{1}{2}\left[\left(\frac{p+1}{2}\right)^2 + 1\right]a_1 + \frac{1}{2}\left[\left(\frac{p+1}{2}\right)^2 - 1\right]a_2,$$

$$(19) \quad g\left(\frac{p+1}{2}b_2\right) = \frac{1}{2}\left[\left(\frac{p+1}{2}\right)^2 - 1\right]a_1 + \frac{1}{2}\left[\left(\frac{p+1}{2}\right)^2 + 1\right]a_2.$$

So $g[(p+1)b_1] = g[(p+1)b_2] =_{(16),(18),(19)} \left[\left(\frac{p+1}{2}\right)^2 + 1\right]a_1 + \left[\left(\frac{p+1}{2}\right)^2 - 1\right]a_2 + \left[\left(\frac{p+1}{2}\right)^2 - 1\right]a_1 + \left[\left(\frac{p+1}{2}\right)^2 + 1\right]a_2 = 2\left(\frac{p+1}{2}\right)^2 a_1 + 2\left(\frac{p+1}{2}\right)^2 a_2 = \frac{1}{2}(p+1)^2 \cdot (a_1 + a_2)$, and $(11)_{p+1}$ holds again. In the total, $(11)_{p+1}$ holds in every case.

We now turn to $(12)_{p+2}$, $(13)_{p+2}$; notice that $p+2$ is odd. By (3) and evenness of g we obtain

$$(20) \quad \begin{aligned} g[(p+1)b_1 + b_2] &= g[(p+1)b_1] + g(b_2), \\ g[b_1 - (p+1)b_2] &= g(b_1) + g[(p+1)b_2]. \end{aligned}$$

Taking into account $(p+1)b_1 + b_2 \perp b_1 - (p+1)b_2$, we infer $g[(p+2)b_1] + g(pb_2) =_{(3)} g[(p+2)b_1 - pb_2] = g[(p+1)b_1 + b_2 + b_1 - (p+1)b_2] =_{(*)} g[(p+1)b_1 + b_2] + g[b_1 - (p+1)b_2] =_{(20)} g[(p+1)b_1] + g(b_2) + g(b_1) + g[(p+1)b_2]$, i.e., by $(13)_p$, (15), $(11)_{p+1}$: $g[(p+2)b_1] + \frac{1}{2}(p^2 - 1)a_1 + \frac{1}{2}(p^2 + 1)a_2 = (p+1)^2(a_1 + a_2) + a_1 + a_2$, i.e., $g[(p+2)b_1] = [(p+1)^2 + 1 - \frac{1}{2}(p^2 - 1)]a_1 + [(p+1)^2 + 1 - \frac{1}{2}(p^2 + 1)]a_2 = \frac{1}{2}[(p+2)^2 + 1]a_1 + \frac{1}{2}[(p+2)^2 - 1]a_2$, which means that $(12)_{p+2}$ holds. $(13)_{p+2}$ is obtained in a similar way. Therefore, (10), (11), (12), (13) do hold.

ii) Conversely, assume that $a_1, a_2 \in Y$ and $g : X \rightarrow Y$ are such that (10), (11), (12), (13) hold. Then g obviously is even. $p = 1$ in (12), (13) gives

$$(21) \quad g(b_1) = a_1, \quad g(b_2) = a_2.$$

Let be $x, z \in X$ arbitrary, but fixed for the moment, $x \perp z$, say $x = p_1b_1 + p_2b_2, z = q_1b_1 + q_2b_2$. By Definition 4

$$(22) \quad p_1q_1 + p_2q_2 = \langle x, z \rangle = 0,$$

hence

$$(23) \quad (p_1 + q_1)^2 + (p_2 + q_2)^2 = p_1^2 + q_1^2 + p_2^2 + q_2^2.$$

By (22) the following cases of parity constellations for p_1, q_1, p_2, q_2 are excluded: Three numbers odd, one number even (4 cases); p_1 and q_1 odd, p_2 and q_2 even, or conversely (2 cases). On the basis of (23), we find by inspection in the ten remaining cases that always

$$g[(p_1 + q_1)b_1] + g[(p_2 + q_2)b_2] = g(p_1b_1) + g(p_2b_2) + g(q_1b_1) + g(q_2b_2)$$

is valid, so by (10) $g(x + z) = g(x) + g(z)$. As $x, z \in X$ with $x \perp z$ were arbitrary, we got $g \in \text{Hom}_\perp(X, Y)$, and in the total $g \in (\text{e}) \text{Hom}_\perp(X, Y)$.

Remark 16. a) The procedure in part i) of the proof of Theorem 15 is different from that in the vector space case. There we have the conclusion

$$(24) \quad g \in (\text{e}) \text{Hom}_\perp(X, Y); x, z \in X; x + z \perp x - z \implies g(x) = g(z)$$

([10], p. 39, Theorem 6, iii). Here (24) is not available in general since the choice $a_1 \neq a_2$, possible by Theorem 15 if Y allows it, leads to $g(b_1) \neq g(b_2)$ although $b_1 + b_2 \perp b_1 - b_2$ (Lemma 5d)). On the other hand, (11) is a weak substitute of (24), but (11) is restricted to p even, and the difference has its origin in the missing 2-divisibility of the free Z -module X .

b) In the vector space case we have $(\text{e}) \text{Hom}_\perp(X, Y) \subset \text{Quad}(X, Y)$ for X at least 2-dimensional ([10], p. 39, Theorem 6). In the context of the present Theorem 15, a mapping $g \in (\text{e}) \text{Hom}_\perp(X, Y)$ is quadratic if and only if $2a_1 = 2a_2$, i.e., if and only if $\omega(a_1) = \omega(a_2)$; if ω is injective, this is equivalent to $a_1 = a_2$. In fact: i) Let $g \in \text{Quad}(X, Y)$. By (11), $g(0) = 0$, and $x = z$ in (JvN) gives $g(2x) = 4g(x)$ ($\forall x \in X$). So $2(a_1 + a_2) =_{(11)} g(2b_1) = 4g(b_1) =_{(12)} 4a_1$, i.e., $2a_2 = 2a_1$. — ii) Let be $2a_1 = 2a_2$. If p is even, then $4 \mid p^2$, so $\frac{1}{2}p^2(a_1 + a_2) = \frac{1}{4}p^2(2a_1 + 2a_2) = p^2a_1 = p^2a_2$. If p is odd, then $4 \mid (p^2 - 1)$, and $\frac{1}{2}(p^2 + 1)a_1 + \frac{1}{2}(p^2 - 1)a_2 = \frac{1}{2}(p^2 - 1)(a_1 + a_2) + a_1 = \frac{1}{4}(p^2 - 1)(2a_1 + 2a_2) + a_1 = (p^2 - 1)a_1 + a_1 = p^2a_1$, and in an analogous way we obtain $\frac{1}{2}(p^2 - 1)a_1 + \frac{1}{2}(p^2 + 1)a_2 = p^2a_2$. So by (10), (11), (12); (13) $g(p_1b_1 + p_2b_2) = g(p_1b_1) + g(p_2b_2) = p_1^2a_1 + p_2^2a_2$ ($\forall p_1, p_2 \in Z$), consequently $g \in \text{Quad}(X, Y)$.

c) Part b) shows how to construct non-quadratic $g \in (\text{e})\text{Hom}_\perp(Z^2, Y)$. Such a g cannot be extended to a $\widehat{g} \in (\text{e})\text{Hom}_\perp(R^2, Y)$, R^2 being equipped with the standard inner product, for g would have to be quadratic as a restriction of a quadratic mapping $\widehat{g}$.

d) It turns out that Theorem 15 describes an exceptional case because we have:

Theorem 17. *If $\dim_Z X \geq 3$, then $g : X \rightarrow Y$ belongs to $(\text{e})\text{Hom}_\perp(X, Y)$ if and only if there exist elements $a_j \in Y$ ($j \in J$) with $2a_j = 2a_k$ for all $j, k \in J$ and $g(x) = \sum p_j^2 a_j$ for all $x = \sum p_j b_j \in X$.*

PROOF. i) Let be $g \in (\text{e})\text{Hom}_\perp(X, Y)$. Put

$$(25) \quad a_j := g(b_j) \quad (\forall j \in J).$$

Let be $j, k \in J$ with $j \neq k$. Since $\dim_Z X \geq 3$, there exists $l \in J$ such that $l \neq j, l \neq k$. By (4), $g(2b_l) = 2g(b_l) + 2g(b_j)$ as well as $g(2b_l) = 2g(b_l) + 2g(b_k)$, so by (25)

$$(26) \quad 2a_j = 2g(b_j) = 2g(b_k) = 2a_k,$$

and this trivially holds also for $j = k$. Now we show by induction on p that

$$(27) \quad (Q, j, p) : \quad g(pb_j) = p^2 a_j \text{ holds for all } j \in J \text{ and all } p \in Z.$$

Evenness of g implies that we need only consider $p \in N^0$. $g(0) = 0$ (ensured by Lemma 8a)) and (25) guarantee that $(Q, j, 0)$ and $(Q, j, 1)$ hold for all $j \in J$.

Let be $n \in N, n \geq 2$, and assume that (Q, j, p) holds for all $p \in N^0$ such that $p \leq n-1$ and for all $j \in J$. Let be $j, k \in J, j \neq k$, arbitrary.

$$\begin{aligned} g[nb_j + (n-2)b_k] &= g[(n-1)(b_j + b_k) + (b_j - b_k)] =_{\text{L.5d}} \\ (28) \quad &= g[(n-1)(b_j + b_k)] + g(b_j - b_k) =_{(3)} \\ &= g[(n-1)b_j] + g[(n-1)b_k] + g(b_j) + g(b_k) \\ &=_{(Q,j,n-1),(Q,k,n-1)} (n-1)^2 g(b_j) + (n-1)^2 g(b_k) + g(b_j) + g(b_k) =_{(25)} \\ &= [(n-1)^2 + 1](a_j + a_k). \end{aligned}$$

On the other hand,

$$(29) \quad \begin{aligned} g[nb_j + (n-2)b_k] &=_{(3)} g(nb_j) + g[(n-2)b_k] =_{(Q,k,n-2)} \\ &= g(nb_j) + (n-2)^2 g(b_k) =_{(25)} g(nb_j) + (n-2)^2 a_k. \end{aligned}$$

From (28) and (29) we obtain $g(nb_j) = [(n-1)^2 + 1]a_j + [(n-1)^2 + 1 - (n-2)^2]a_k = (n^2 - 2n + 2)a_j + (2n-2)a_k =_{(26)} (n^2 - 2n + 2)a_j + (2n-2)a_j = n^2 a_j$. Analogously, with j, k interchanged, we obtain also $g(nb_k) = n^2 a_k$. So (Q, j, n) holds for all $j \in J$, and by induction, (27) is established.

If $x = \sum p_j b_j \in X$ is arbitrary, then $g(x) =_{(3)} \sum g(p_j b_j) =_{(27)} \sum p_j^2 a_j$, i.e., g has the form required.

ii) Assume that there exist $a_j \in Y (j \in J)$ such that $2a_j = 2a_k (\forall j, k \in J)$ and $g(x) = \sum p_j^2 a_j$ for all $x = \sum p_j b_j \in X$. Let be $x, z \in X$, $x \perp z$, say $x = \sum p_j b_j$, $z = \sum q_j b_j$, so $x + z = \sum (p_j + q_j) b_j$. By Definition 4

$$(30) \quad \sum p_j q_j = \langle x, z \rangle = 0$$

Let d be the common value of all elements $2a_j (j \in J)$. Then $g(x+z) = \sum (p_j + q_j)^2 a_j = \sum (p_j^2 + 2p_j q_j + q_j^2) a_j = \sum p_j^2 a_j + \sum p_j q_j \cdot 2a_j + \sum q_j^2 a_j = g(x) + \sum p_j q_j \cdot d + g(z) =_{(30)} g(x) + g(z)$. So $g \in \text{Hom}_\perp(X, Y)$, and obviously g is even.

Remark 18. a) The mappings $g : X \rightarrow Y$ of the form $g(x) = \sum p_j^2 a_j$ for all $x = \sum p_j b_j \in X$, as occurring in Theorem 17, are quadratic no matter whether $2a_j = 2a_k (\forall j, k \in J)$ or not. In fact: If also $z = \sum q_j b_j \in X$, then $g(x+z) + g(x-z) = \sum (p_j + q_j)^2 a_j + \sum (p_j - q_j)^2 a_j = \sum [(p_j + q_j)^2 + (p_j - q_j)^2] a_j = \sum (2p_j^2 + 2q_j^2) a_j = 2 \sum p_j^2 a_j + 2 \sum q_j^2 a_j = 2g(x) + 2g(z)$.

b) A quadratic mapping $g : X \rightarrow Z$ satisfying $2g(b_j) = 2g(b_k) (\forall j, k \in J)$ need not be in $(e) \text{Hom}_\perp(X, Z)$. This shows that the quadratic mappings occurring in Theorem 17 form a very special class in $\text{Quad}(X, Y)$. In fact: $g(x) := (\sum p_j)^2$ for $x = \sum p_j b_j \in X$. For $x, z \in X$ we have $g(x+z) + g(x-z) = (\sum (p_j + q_j))^2 + (\sum (p_j - q_j))^2 = (\sum p_j + \sum q_j)^2 + (\sum p_j - \sum q_j)^2 = 2(\sum p_j)^2 + 2(\sum q_j)^2 = 2g(x) + 2g(z)$, i.e., $g \in \text{Quad}(X, Z)$. Furthermore $g(b_j) = 1 (\forall j \in J)$. If $\dim_Z X \geq 2$ and $b_j \perp b_k$, then $g(b_j + b_k) = 2^2 \neq 2 = g(b_j) + g(b_k)$, i.e., $g \notin (e) \text{Hom}_\perp(X, Z)$.

Remark 19. Theorems 13 and 17 and Remark 18a) show that for sufficiently large dimension of X ,

$$(31) \quad (\text{o}) \operatorname{Hom}_{\perp}(X, Y) = \operatorname{Hom}(X, Y),$$

$$(32) \quad (\text{e}) \operatorname{Hom}_{\perp}(X, Y) \subset \operatorname{Quad}(X, Y),$$

and Remark 18b) shows that the inclusion in (32) may be strict. Examples 10 and 11 demonstrate that (31), (32) do not hold for $\dim_Z X = 1$. Theorem 15 exhibits the exceptional case for $(\text{e}) \operatorname{Hom}_{\perp}(X, Y)$ when $\dim_Z X = 2$ where (32) still can be violated for suitable groups Y . The proof of Theorem 17 shows in what way the hypothesis $\dim_Z X \geq 3$ enforces the condition (26) $2a_j = 2a_k (\forall j, k \in J)$ and then the quadratic character of the mapping g considered there (for the 2-dimensional situation cf. Remark 16b).

There are many situations in connection with inner product spaces for which dimension 2 is exceptional. The most prominent one might be the characterization of inner product spaces among normed spaces where certain conditions are effective only for $\dim X \geq 3$ ([2], p. 97–156). For a much simpler question concerning a characterization of the inner product on real and complex vector spaces cf. [9].

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