

On rank of apparition of primes in Lucas sequences

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Let $R = \{R_n\}_{n=0}^\infty$ be a sequence of Lucas numbers defined by the recursion

$$R_n = A \cdot R_{n-1} + B \cdot R_{n-2},$$

where A and B are fixed non-zero integers and the initial terms are $R_0=0$, $R_1=1$. Denote the roots of the characteristic polynomial $x^2 - Ax - B$ of the sequence by α and β . Throughout this paper we assume that R is a non-degenerated sequence, i.e. α/β is not a root of unity.

If p is a prime and $p \nmid B$ then, as it is known, there are terms in the sequence divisible by p . If $p \mid R_n$ but $p \nmid R_m$ for $0 < m < n$, then we say that n is the rank of the apparition of p in the sequence and we denote it by $r(p)$. We also say that p is a primitive prime divisor of R_n if $r(p)=n$. We know that R_n has at least one primitive prime divisor for any $n > n_0$, where n_0 is an absolute constant (see e.g. C. L. STEWART [4]) and $r(p)$ is a divisor of $p - (D/p)$ for any prime with $p \nmid B$, where $D = A^2 + 4B$ and (D/p) is the Legendre's symbol with $(D/p)=0$ in case $p \mid D$ (see e.g. D. H. LEHMER [3]). Thus the primitive prime divisors of a term R_n are of the form $p = nk \pm 1$ but we do not know how large k is in general or how small can be the ratios $r(p)/p$.

It is clear that $r(p)/p \leq 1 + 1/p$, but from a result of [2] it follows that $r(p)/p$ can be arbitrarily small. We note that this also follows from a result of D. JARDEN ([1], p. 5) if R is the Fibonacci sequence. In this paper we give bounds for the average order of the ratios $r(p)/p$ which yield some results on the primitive prime divisors of Lucas numbers.

In the results and in their proofs $c_1, c_2, \dots, x_1, x_2, \dots$ will denote positive constants which are absolute ones or they depend only on the sequence R . Furthermore we assume $r(p)=0$ if $p \mid B$.

Theorem 1. *There are positive constants c_1 and c_2 such that*

$$c_1 \cdot \frac{\sqrt{x}}{\log x} < \sum_{p \leq x} \frac{r(p)}{p} < c_2 \cdot \frac{x}{\log x}$$

for any $x > x_1$.

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Theorem 2. *There are positive constants c_3 and c_4 such that*

$$c_3 \cdot \frac{\sqrt{x}}{\log x} < \sum_{\substack{p \\ r(p) \equiv x}} \frac{r(p)}{p} < c_4 \cdot x$$

for any $x > x_2$.

The magnitude of prime and primitive prime divisors of Lucas numbers was investigated in many papers (see C. L. STEWART [5] and its exhaustive references). Among others it was proved that for almost all natural numbers n the greatest primitive prime divisor of R_n is greater than $\varepsilon(n) \cdot n \cdot (\log n)^2 / \log \log n$, where $\varepsilon(n)$ is any real valued function for which $\varepsilon(n) \rightarrow 0$ as $n \rightarrow \infty$. The following corollary shows that n^2 is an upper bound for infinitely many primitive prime divisors.

Corollary 1. *Let x and c be real numbers with the conditions $0 < c < 1/2$ and $x > x(c, R)$, where $x(c, R)$ depends only on c and the sequence R . Then there are at least x^c primes p for which $p \leq x$ and $p < (r(p))^{1/c}$.*

It is known that $|R_n| < e^{c_5 n}$ for any $n > 0$ and, as we have seen, $p \leq n-1$ if $r(p) = n$. From these it follows that R_n has at most $c_6 n / \log n$ primitive prime divisors and the trivial estimation

$$\sum_{\substack{p \\ r(p)=n}} \frac{1}{p} < \frac{c_7}{\log n}$$

holds for any $n > 1$. From Theorem 2 a better estimation follows for almost all n which shows a connection between the magnitude and the number of primitive prime divisors of the Lucas numbers.

Corollary 2. *For any $\delta > 0$*

$$\sum_{\substack{p \\ r(p)=n}} \frac{1}{p} < \frac{\delta \cdot \log n}{n}$$

for almost all n , i.e. for all natural numbers n except perhaps for those in a set of asymptotic density zero.

Now we prove our results.

PROOF OF THEOREM 1. The right inequality trivially holds because

$$\frac{r(p)}{p} \equiv \frac{p+1}{p} = 1 + \frac{1}{p}$$

for any p and

$$\sum_{p \leq x} \frac{1}{p} = \mathcal{O}(\log \log x).$$

Let $a(n)$ be an arithmetical function such that $a(n) = r(n)$ if n is a prime number and $a(n) = 0$ otherwise. Then for any sufficiently large x

$$(1) \quad S_x = \sum_{p \leq x} \frac{r(p)}{p} = \sum_{n \leq x} a(n) \cdot \frac{1}{n}.$$

Before we estimate the sum S_x we give a lower estimation for the sum

$$A(x) = \sum_{n \leq x} a(n).$$

There are $\pi(x) - c_8$ summands in $A(x)$ different from zero and, as we have seen above, there are at most $c_6 n / \log n$ primes for which $r(p) = n$. But, using Euler's summation formula,

$$\sum_{n \leq c_9 \sqrt{x}} \frac{c_6 n}{\log n} = c_6 \cdot \int_2^{c_9 \sqrt{x}} \frac{t}{\log t} dt + \mathcal{O}\left(\frac{\sqrt{x}}{\log x}\right) < c_{10} \cdot \frac{x}{\log x},$$

where c_9 can be chosen such that

$$c_{10} \cdot \frac{x}{\log x} < \pi(x) - c_8,$$

and so

$$(2) \quad A(x) > \sum_{n \leq c_9 \sqrt{x}} n \cdot \frac{c_6 n}{\log n} = c_6 \cdot \int_2^{c_9 \sqrt{x}} \frac{t^2}{\log t} dt + \mathcal{O}\left(\frac{x}{\log x}\right) > c_{11} \cdot \frac{x \cdot \sqrt{x}}{\log x}.$$

Using Abel's identity, by (1) and (2)

$$\begin{aligned} S_x &= \frac{A(x)}{x} + \int_2^x \frac{A(t)}{t^2} dt + \mathcal{O}(1) > \\ &> c_{11} \cdot \frac{\sqrt{x}}{\log x} + c_{11} \cdot \int_2^x \frac{dt}{\sqrt{t} \cdot \log t} + \mathcal{O}(1) > c_{12} \cdot \frac{\sqrt{x}}{\log x} \end{aligned}$$

follows which proves the left inequality of the theorem.

PROOF OF THEOREM 2. The left inequality follows from Theorem 1 since $r(p) \leq x$ for any prime for which $p \leq x-1$.

Let x be a sufficiently large number and let p be a prime such that $p \leq x$ and $r(p) = n \leq x$. Then there is an integer k such that $p = kn+1$ or $p = kn-1$ and $k \leq \frac{x+1}{n}$ and so, using that $\sum_{k \leq t} 1/k = \log t + \mathcal{O}(1)$,

$$\begin{aligned} (3) \quad \sum_{\substack{p \\ r(p)=n}} \frac{1}{p} &\leq \frac{1}{n-1} + \frac{1}{n+1} + \frac{1}{2n-1} + \frac{1}{2n+1} + \dots = \\ &= 2 \cdot \sum_{k \leq (x+1)/2n} \frac{1}{kn} + \mathcal{O}\left(\frac{1}{n}\right) = \frac{2}{n} (\log x - \log n) + \mathcal{O}\left(\frac{1}{n}\right) \end{aligned}$$

follows. Since

$$\sum_{\substack{p \\ r(p) \leq x}} \frac{r(p)}{p} = \sum_{n \leq x} \left(n \cdot \sum_{\substack{p \\ r(p)=n}} \frac{1}{p} \right),$$

by (3) we have

$$(4) \quad \sum_{\substack{p \\ r(p) \leq x}} \frac{r(p)}{p} \leq 2 \cdot \sum_{n \leq x} (\log x - \log n) + \sum_{\substack{p > x \\ r(p) \leq x}} \frac{r(p)}{p} + \mathcal{O}(x).$$

We have seen that there are at most $c_6 n / \log n$ primes p for which $r(p) = n$, furthermore

$$\sum_{n \leq x} \frac{c_6 n}{\log n} < c_{13} \cdot \frac{x^2}{\log x}$$

and there are at most $c_{13} x^2 / \log x$ primes not exceeding $c_{14} x^2$, therefore

$$(5) \quad \sum_{\substack{p > x \\ r(p) \leq x}} \frac{r(p)}{p} \leq x \cdot \sum_{\substack{p > x \\ r(p) \leq x}} \frac{1}{p} \leq x \cdot \left(\sum_{p \leq c_{14} x^2} \frac{1}{p} - \sum_{p \leq x} \frac{1}{p} \right) = \mathcal{O}(x).$$

On the other hand

$$(6) \quad \sum_{n \leq x} \log x = x \cdot \log x + \mathcal{O}(\log x)$$

and

$$(7) \quad \sum_{n \leq x} \log n = x \cdot \log x - x + \mathcal{O}(\log x)$$

and so (4), by (5), (6) and (7), implies the right hand inequality of the theorem which completes the proof.

PROOF OF COROLLARY 1. Suppose that there are at most x^c primes p such that $p \leq x$ and

$$\frac{r(p)}{p} > \frac{1}{p^{1-c}}.$$

Then, using that $r(p) \leq p+1$ and so $r(p)/p \leq 1 + \frac{1}{p}$ for any prime, with the notation $y = \pi(x) - x^c$ we have

$$(8) \quad \sum_{p \leq x} \frac{r(p)}{p} < \sum_{n \leq y} \frac{1}{p_n^{1-c}} + x^c + \sum_{n \leq x^c} \frac{1}{p_n} < \sum_{p \leq x} \frac{1}{p^{1-c}} + x^c + \mathcal{O}(\log \log x),$$

where p_n is the n^{th} prime. But

$$\sum_{p \leq x} \frac{1}{p^{1-c}} = \mathcal{O}\left(\frac{x^c}{\log x}\right)$$

and so (8) contradicts Theorem 1 if x is sufficiently large since $c < 1/2$.

Thus there are at least x^c prime numbers p not exceeding x for which

$$(9) \quad \frac{r(p)}{p} > \frac{1}{p^{1-c}}.$$

For these primes by (9)

$$p < (r(p))^{1/c}$$

which proves the assertion.

PROOF OF COROLLARY 2. Let x and δ be positive numbers and let N_x be the set of natural numbers n for which $n \leq x$ and

$$\sum_{\substack{p \\ r(p)=n}} \frac{1}{p} \geq \frac{\delta \cdot \log n}{n}.$$

If $|N_x| = \varepsilon x$, where $|N_x|$ denotes the cardinality of the set N_x , then

$$\begin{aligned} \sum_{\substack{p \\ r(p) \leq x}} \frac{r(p)}{p} &= \sum_{n \leq x} \left(n \cdot \sum_{\substack{p \\ r(p)=n}} \frac{1}{p} \right) \cong \sum_{n \leq \varepsilon x} \delta \cdot \log n = \\ &= \delta \cdot (\varepsilon x \cdot \log \varepsilon x - \varepsilon x + \mathcal{O}(\log x)) \end{aligned}$$

which does not contradict Theorem 2 only if $\varepsilon \rightarrow 0$ as $x \rightarrow \infty$. From this the corollary follows.

References

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