

Differences

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Notations and preliminary results

Let $\mathbf{R}$ be the set of real numbers and let $\mathbf{R}_+$ be the set of positive real numbers. A function $\alpha: \mathbf{R}_+ \rightarrow \mathbf{R}$ (or $\alpha: \mathbf{R} \rightarrow \mathbf{R}$) is a Jensen-function if

$$(1) \quad 2\alpha\left(\frac{x+y}{2}\right) = \alpha(x) + \alpha(y)$$

is valid for all $x, y \in \mathbf{R}_+$ (or for all $x, y \in \mathbf{R}$).

Denote by $J(\mathbf{R}_+ \rightarrow \mathbf{R})$ and $J(\mathbf{R} \rightarrow \mathbf{R})$ the class of Jensen-functions on $\mathbf{R}_+$ and $\mathbf{R}$, respectively.

The following theorem was proved by Z. DARÓCZY, L. SZÉKELYHIDI and myself in [1]:

Theorem 1. *Let $f: \mathbf{R}_+ \rightarrow \mathbf{R}$ be a function, such that the function $\Delta_\lambda f: \mathbf{R}_+ \rightarrow \mathbf{R}$ defined by the difference*

$$(2) \quad \Delta_\lambda f(x) = f(\lambda x) - f(x) \quad (x \in \mathbf{R}_+)$$

is a Jensen-function (i.e. $\Delta_\lambda f \in J(\mathbf{R}_+ \rightarrow \mathbf{R})$) for all fixed $\lambda \in \mathbf{R}_+$. Then f has the form

$$f(x) = \alpha(x) + m(x) \quad (x \in \mathbf{R}_+),$$

where $\alpha \in J(\mathbf{R}_+ \rightarrow \mathbf{R})$ and $m: \mathbf{R}_+ \rightarrow \mathbf{R}$ satisfies the Cauchy functional equation

$$m(xy) = m(x) + m(y) \quad (x, y \in \mathbf{R}_+).$$

This theorem and its generalizations have applications to find the general solution of some functional equations (see [1], [2], [3]).

In this paper we consider the difference

$$\Delta_\lambda^+ f(x) = f(x + \lambda) - f(x) \quad (x, \lambda \in \mathbf{R})$$

instead of (2) and we prove a theorem analogous to Theorem 1.

2. The main result

Theorem 2. Let $f: \mathbf{R} \rightarrow \mathbf{R}$ be an arbitrary function, such that the function

$$(3) \quad \Delta_{\lambda}^+ f: \mathbf{R} \rightarrow \mathbf{R}, \quad \Delta_{\lambda}^+ f(x) = f(x+\lambda) - f(x) \quad (x \in \mathbf{R})$$

is a Jensen-function for all fixed $\lambda \in \mathbf{R}$. Then f has the form

$$(4) \quad f(x) = \alpha(x) + N(x) \quad (x \in \mathbf{R}),$$

where $\alpha \in J(\mathbf{R} \rightarrow \mathbf{R})$ and $N: \mathbf{R} \rightarrow \mathbf{R}$ satisfies the functional equation

$$(5) \quad N(x+y) + N(x-y) = 2N(x) + 2N(y) \quad (x, y \in \mathbf{R}).$$

PROOF. If $\Delta_{\lambda}^+ f \in J(\mathbf{R} \rightarrow \mathbf{R})$ for all fixed $\lambda \in \mathbf{R}$, then the function $f: \mathbf{R} \rightarrow \mathbf{R}$ satisfies the functional equation

$$(6) \quad 2f\left(\frac{x+y}{2} + \lambda\right) - 2f\left(\frac{x+y}{2}\right) = f(x+\lambda) - f(x) + f(y+\lambda) - f(y) \quad (x, y, \lambda \in \mathbf{R}).$$

In (6) let us replace $-x$, $-y$ and $-\lambda$ by x , y and λ respectively, then we get

$$(7) \quad 2f\left[-\left(\frac{x+y}{2} + \lambda\right)\right] - 2f\left[-\frac{x+y}{2}\right] = \\ = f[-(x+\lambda)] - f(-x) + f[-(y+\lambda)] - f(-y) \quad (x, y, \lambda \in \mathbf{R}).$$

We easily obtain from (6) and (7) that the functions

$$\bar{f}: \mathbf{R} \rightarrow \mathbf{R}, \quad \bar{f}(x) = f(x) - f(-x) \quad (x \in \mathbf{R})$$

and

$$f^*: \mathbf{R} \rightarrow \mathbf{R}, \quad f^*(x) = f(x) + f(-x) \quad (x \in \mathbf{R})$$

satisfy the functional equation (6), moreover $\bar{f}$ is odd and f^* is even.

In addition, we have

$$(8) \quad f(x) = \frac{1}{2} [\bar{f}(x) + f^*(x)] \quad (x \in \mathbf{R}).$$

By substitution $\lambda = -y$ in (6), we get for $\bar{f}$ the functional equation

$$(9) \quad 2\bar{f}\left(\frac{x-y}{2}\right) - 2\bar{f}\left(\frac{x+y}{2}\right) = \bar{f}(x-y) - \bar{f}(x) - \bar{f}(y) \quad (x, y \in \mathbf{R}).$$

Interchanging x and y here, using that $\bar{f}$ is an odd function, we get

$$-2\bar{f}\left(\frac{x-y}{2}\right) - 2\bar{f}\left(\frac{x+y}{2}\right) = -\bar{f}(x-y) - \bar{f}(x) - \bar{f}(y) \quad (x, y \in \mathbf{R}).$$

Adding this equation to (9), we obtain

$$2\bar{f}\left(\frac{x+y}{2}\right) = \bar{f}(x) + \bar{f}(y) \quad (x, y \in \mathbf{R}),$$

i.e. $\bar{f} \in J(\mathbf{R} \rightarrow \mathbf{R})$.

Now consider the functional equation (6) for function f^* and substitute here $\lambda = -y$, then replace y with $-y$. Thus we get

$$(10) \quad 2f^*\left(\frac{x+y}{2}\right) - 2f^*\left(\frac{x-y}{2}\right) = f^*(x+y) - f^*(x) - f^*(y) + f^*(0) \quad (x, y \in \mathbf{R}).$$

With the substitution $x=y=\frac{t}{2}$, we get from (10) that

$$f^*\left(\frac{t}{2}\right) = \frac{1}{4}f^*(t) + \frac{3}{4}f^*(0) \quad (t \in \mathbf{R}).$$

Using this equation, we obtain from (10) the equation

$$f^*(x+y) + f^*(x-y) = 2f^*(x) + 2f^*(y) - 2f^*(0) \quad (x, y \in \mathbf{R}),$$

which implies that the function

$$N: \mathbf{R} \rightarrow \mathbf{R}, \quad N(x) = f^*(x) - f^*(0) \quad (x \in \mathbf{R})$$

satisfies the functional equation (5). Thus f^* has the form

$$(11) \quad f^*(x) = N(x) + f^*(0) \quad (x \in \mathbf{R}),$$

where $f^*(0) \in \mathbf{R}$ is an arbitrary constant.

From (8) and (11) it follows that

$$f(x) = \frac{1}{2}\tilde{f}(x) + \frac{1}{2}f^*(0) + \frac{1}{2}N(x) \quad (x \in \mathbf{R}).$$

Since $\frac{1}{2}\tilde{f} + \frac{1}{2}f^*(0) \doteq \alpha \in J(\mathbf{R} \rightarrow \mathbf{R})$ and $\frac{1}{2}N$ also satisfies the functional equation (5), we obtain the form (4) for f .

Thus the Theorem 2 is proved.

3. Remarks and applications

Remark 1. The converses of Theorems 1 and 2 are also true.

E.g. if f has the form (4), then using the property $4N\left(\frac{x}{2}\right) = N(x)$ ($x \in \mathbf{R}$) of the function N , we have

$$\begin{aligned} 2\Delta_{\lambda}^+ f\left(\frac{x+y}{2}\right) &= 2\alpha\left(\frac{x+y}{2} + \lambda\right) - 2\alpha\left(\frac{x+y}{2}\right) + 2N\left(\frac{x+y}{2} + \lambda\right) - 2N\left(\frac{x+y}{2}\right) = \\ &= 2\alpha\left(\frac{(x+\lambda) + (y+\lambda)}{2}\right) - 2\alpha\left(\frac{x+y}{2}\right) + 2N\left(\frac{x+\lambda}{2} + \frac{y+\lambda}{2}\right) - 2N\left(\frac{x}{2} + \frac{y}{2}\right) = \end{aligned}$$

$$\begin{aligned}
&= \alpha(x+\lambda) + \alpha(y+\lambda) - \alpha(x) - \alpha(y) - 2N\left(\frac{x+\lambda}{2} - \frac{y+\lambda}{2}\right) + 4N\left(\frac{x+\lambda}{2}\right) + 4\left(\frac{y+\lambda}{2}\right) + \\
&\quad + 2N\left(\frac{x-y}{2}\right) - 4N\left(\frac{x}{2}\right) - 4N\left(\frac{y}{2}\right) = \alpha(x+\lambda) - \alpha(x) + \alpha(y+\lambda) - \alpha(y) + \\
&\quad + N(x+\lambda) - N(x) + N(y+\lambda) - N(y) = \Delta_{\lambda}^{+} f(x) + \Delta_{\lambda}^{+} f(y),
\end{aligned}$$

i.e. $\Delta_{\lambda}^{+} f \in J(\mathbf{R} \rightarrow \mathbf{R})$.

Remark 2. A similar theorem was used by GY. MAKSA (see [4]) in connection with the functional equation

$$f(x+y) + g(xy) = h(x) + h(y) \quad (x, y \in \mathbf{R}_+),$$

which was considered for the first time by Z. DARÓCZY in 1969.

Remark 3. Theorem 2 can be generalized for functions $f: K \rightarrow A$, where K and A are additive Abelian groups.

Remark 4. Theorem 2 can be used to solve e.g. the functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) + g(x-y) \quad (x, y \in \mathbf{R})$$

for the unknown functions $f, g: \mathbf{R} \rightarrow \mathbf{R}$.

References

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- [3] K. LAJKÓ, On the general solution of rectangle-type functional equations, *Publ. Math. (Debrecen)*, **28** (1981), 137—143.
- [4] GY. MAKSA, On the functional equation $f(x+y) + g(xy) = h(x) + h(y)$, *Publ. Math. (Debrecen)*, **24** (1977), 25—29.

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