Lacunary spline interpolation in L^p

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

Recently in [3] TH. FAWZY and F. HOLAIL have presented a (0, 2)-type lacunary interpolation method by splines of degree 6. Here we show that the same method can be applied for approximation in L^p, if the function values and the second derivatives are replaced by integral mean values and second order difference quotients, respectively. Here we present our results in $L^p(\mathbf{R})$, but we remark, that similar results can be proved in the case of $L^p(a, b)$, where (a, b) is any finite open interval. Our main ideas are similar to those of [4].

For any integer $r \geqq 0$ the space $\mathbf{W}_p^r$ with $1 \leqq p < \infty$ consists of all r-times differentiable functions with an r-th derivative belonging to L^p. The L^p-modulus of continuity $\omega_r(f, h)_p$ for any f in L^p is defined as usually $(h>0)$ (see e.g. [1], [2]).

The following theorem — which is a slight modification of a result in [1] — has been proved in [4].

Lemma. *Let Γ be an arbitrary set and L_γ: $L^p \to L^p$ $(1 \leqq p < \infty)$ for any γ in Γ, uniformly bounded linear operators, for which there exist a function a: $\Gamma \to [0, 1]$, an integer $r \geqq 1$ and a constant $c>0$ with the property that*

$$\|L_\gamma(f)-f\|_p \leqq ca(\gamma)\|f^{(r)}\|_p$$

whenever f is in $\mathbf{W}_p^r$. Then there exists a constant $d>0$ with

$$\|L_\gamma(f)-f\|_p \leqq d\omega_r\big(f, a(\gamma)^{1/r}\big)_p$$

whenever f is in L^p.

Let $f \in L^p$ be arbitrary and we define

$$f_h(x) = \frac{1}{h} \int_{x-h/2}^{x+h/2} f(t)\, dt.$$

The function f_h is called the Steklov-transform of f. Let $h>0$ and $\{x_k\}$ be a subdivision of $\mathbf{R}$ with $x_{k+1}-x_k=h$. Let for all k

$$f_k = f_h(x_k), \quad f_k'' = \frac{1}{h^2} \cdot \Delta^2 f_k$$

and

$$S_h(x) = \sum_{j=0}^{6} \frac{S_k^{(j)}}{j!} \cdot (x-x_k)^j, \qquad x_k \leqq x \leqq x_{k+1}$$

where

$$S_k^{(0)} = f_k, \quad S_k^{(2)} = f_k'',$$

$$S_k^{(6)} = \frac{1}{h^4} \cdot \Delta^4 f_{k-1}'',$$

$$S_k^{(5)} = \frac{1}{h^3} \cdot \Delta^3 f_{k-1}'' - \frac{h}{2} S_k^{(6)},$$

$$S_k^{(4)} = \frac{1}{h^2} \cdot \Delta^2 f_{k-1}'' - \frac{h^2}{12} S_k^{(6)},$$

$$S_k^{(3)} = \frac{1}{h} \cdot \Delta f_k - \sum_{r=2}^{4} \frac{h^{r-1}}{r!} S_k^{(r+2)},$$

$$S_k^{(1)} = \frac{1}{h} \cdot \Delta f_k - \frac{h}{2} f_k'' - \sum_{r=3}^{6} \frac{h^{r-1}}{r!} S_k^{(r)}.$$

Then this S_h is a continuous function with a right continuous second derivative, and it is a piecewise polynomial of degree 6 (see [3]).

Theorem. *For any f in L^p $(1 \leqq p < \infty)$ we have*

$$\|S_h - f\|_p \leqq \text{const}\, \omega_2(f, h)_p$$

for $0 < h \leqq 1$.

Proof. It is trivial that the operator $f \to S_h$ is linear. In order to apply the lemma we first prove the uniform boundedness.

In what follows we shall often use the inequality

$$\Big|\int_E f\Big|^p \leqq m(E)^{p-1} \int_E |f|^p, \qquad (f \in L^p(E),\ 1 \leqq p < \infty).$$

We have

$$\|S_h\|_p \leqq \sum_{j=0}^{6} \Big[\sum_k \int_{x_k}^{x_{k+1}} \Big|\frac{S_k^{(j)}}{j!} (x-x_k)^j\Big|^p \, dx\Big]^{1/p} =$$

$$= \sum_{j=0}^{6} \frac{h^{j+1/p}}{j!(pj+1)^{1/p}} \big[\sum_k |S_k^{(j)}|^p\big]^{1/p}.$$

As $S_k^{(j)}$ is a linear combination of terms of the type $\frac{1}{h^j} f_k$, we first estimate the quantity

$$\Big[\Big|\frac{1}{h^j} f_k\Big|^p\Big]^{1/p} = \frac{1}{h^j} \Big[\sum_k \Big|\frac{1}{h} \cdot \int_{x_k - h/2}^{x_k + h/2} f(x)\, dx\Big|^p\Big]^{1/p} \leqq$$

$$\leqq \frac{1}{h^j} \Big[\sum_k \frac{1}{h^p} h^{p-1} \int_{x_k - h/2}^{x_k + h/2} |f(x)|^p \, dx\Big]^{1/p} = h^{-j-1/p} \|f\|_p$$

and, by the above estimation, we get

$$\|S_h\|_p \leqq \text{const}\, \|f\|_p$$

which proves the uniform boundedness.

Now let $f \in \mathbf{W}_p^2$. By the Taylor-formula we have

$$f(x) = f(x_k) + f'(x_k)(x-x_k) + \int_{x_k}^{x} (x-\xi) f''(\xi)\, d\xi$$

where $x_k < \xi < x$ and hence for all $x_k \leqq x \leqq x_{k+1}$

$$\text{(1)} \qquad S_h(x) - f(x) = [f_k - f(x_k)] + \left[\left(\frac{1}{h}\cdot \Delta f_k - f'(x_k)\right)(x-x_k)\right] -$$

$$-\left[\frac{h}{2} f_k''(x-x_k)\right] - \left[\sum_{r=3}^{6} \frac{h^{r-1}}{r!} S_k^{(r)}(x-x_k)\right] + \left[\frac{1}{2} f_k''(x-x_k)^2\right] - \left[\int_{x_k}^{x} (x-\xi) f''(\xi)\, d\xi\right] +$$

$$+\left[\frac{1}{3!} S_k^{(3)}(x-x_k)^3\right] + \left[\frac{1}{4!} S_k^{(4)}(x-x_k)^4\right] + \left[\frac{1}{5!} S_k^{(5)}(x-x_k)^5\right] + \left[\frac{1}{6!} S_k^{(6)}(x-x_k)^6\right].$$

It is enough to prove that all terms in the square brackets have L^p-norms which are not greater than const $h^2 \|f''\|_p$.

First of all we estimate the quantity $\left[\sum_k \left|\frac{1}{h^2}\cdot \Delta^2 f_k\right|^p\right]^{1/p}$.

$$\sum_k \left|\frac{1}{h^2}\cdot \Delta^2 f_k\right|^p = \sum_k \left|\frac{1}{h}\left[\frac{f_h(x_{k+2}) - f_h(x_{k+1})}{h} - \frac{f_h(x_{k+1}) - f_h(x_k)}{h}\right]\right|^p =$$

$$= \sum_k \left|\frac{f_h'(\xi_{k+1}) - f_h'(\xi_k)}{h}\right|^p = \sum_k \left|\frac{1}{h}\int_{\xi_k}^{\xi_{k+1}} f_h''(\xi)\, d\xi\right|^p \leqq$$

$$\leqq \frac{1}{h^p} \sum_k (\xi_{k+1} - \xi_k)^{p-1} \int_{x_k}^{x_{k+2}} |f_h''(\xi)|^p\, d\xi \leqq \frac{1}{h^p}(2h)^{p-1} 2 \|f_h''\|_p^p = \frac{2^p}{h} \|f_h''\|_p^p,$$

where $x_k < \xi_k < x_{k+1} < \xi_{k+1} < x_{k+2}$. But

$$\|f_h''\|_p = \left[\int |f_h''(x)|^p\, dx\right]^{1/p} = \left[\int \left|\frac{1}{h} f'\left(x+\frac{h}{2}\right) - \frac{1}{h} f'\left(x-\frac{h}{2}\right)\right|^p dx\right]^{1/p} =$$

$$= \frac{1}{h}\left[\int \left|f'\left(x+\frac{h}{2}\right) - f'\left(x-\frac{h}{2}\right)\right|^p dx\right]^{1/p} = \frac{1}{h}\left[\int \left|\int_{x-h/2}^{x+h/2} f''(t)\, dt\right|^p dx\right]^{1/p} \leqq$$

$$\leqq \frac{1}{h}\left[\int h^{p-1} \int_{x-h/2}^{x+h/2} |f''(t)|^p\, dt\, dx\right]^{1/p} = \frac{1}{h}\left[h^{p-1} \sum_k \int_{x_k}^{x_{k+1}} \int_{x-h/2}^{x+h/2} |f''(t)|^p\, dt\, dx\right]^{1/p} \leqq$$

$$\leqq \frac{1}{h}\left[h^{p-1} \sum_k \int_{x_k}^{x_{k+1}} \int_{x_k-h/2}^{x_{k+1}+h/2} |f''(t)|^p\, dt\, dx\right]^{1/p} = \frac{1}{h}\left[h^p \sum_k \int_{x_k-h/2}^{x_{k+1}+h/2} |f''(t)|^p\, dt\right]^{1/p} =$$

$$= 2\|f''\|_p$$

and hence

$$\left[\sum_k \left|\frac{1}{h^2}\cdot \Delta^2 f_k\right|^p\right]^{1/p} \leqq \frac{4}{h^{1/p}}\|f''\|_p.$$

On the other hand, it is easy to see, that $S_k^{(r)}$ $(r\geqq 3)$ is a linear combination of terms of the type

$$\frac{1}{h^{r-2}}\left[\frac{1}{h^2}\cdot \Delta^2 f_k\right],$$

which implies that for $r\geqq 3$

$$[\sum_k |S_k^{(r)}|^p]^{1/p} \leqq \text{const}\,\frac{1}{h^{r-2+1/p}}\|f''\|_p.$$

From this fact we infer that for $j\geqq 3$

$$\left[\sum_k \int_{x_k}^{x_{k+1}} \left|\frac{1}{j!}S_k^{(j)}(x-x_k)^j\right|^p dx\right]^{1/p} = \frac{1}{j!}\left[\frac{h^{pj+1}}{pj^{+1}}\sum_k |S_k^{(j)}|^p\right]^{1/p} \leqq \text{const}\, h^2\|f''\|_p,$$

which is the desired estimation for the L^p-norm of the last four terms in (1).

For the first term we have

$$[\sum_k \int_{x_k}^{x_{k+1}} |f_k-f(x_k)|^p\,dx]^{1/p} = \left[h\sum_k\left|\frac{1}{h}\int_{x_k-h/2}^{x_k+h/2}[f(x)-f(x_k)]\,dx\right|^p\right]^{1/p} =$$

$$= \left[h\sum_k\left|\frac{1}{h}\int_{x_k-h/2}^{x_k+h/2}\left[f'(x_k)(x-x_k)+\int_{x_k}^{x}(x-\xi)f''(\xi)\,d\xi\right]dx\right|^p\right]^{1/p} =$$

$$= \left[h\sum_k\left|\frac{1}{h}\int_{x_k-h/2}^{x_k+h/2}\int_{x_k}^{x}(x-\xi)f''(\xi)\,d\xi\,dx\right|^p\right]^{1/p} =$$

$$\leqq \left[h^{1-p}\sum_k h^{p-1}\int_{x_k-h/2}^{x_k+h/2}\left[\int_{x_k}^{x}|(x-\xi)f''(\xi)|\,d\xi\right]^p dx\right]^{1/p} \leqq$$

$$\leqq \left[\sum_k h^p\int_{x_k-h/2}^{x_k+h/2}\left[\int_{x_k-h/2}^{x_k+h/2}|f''(\xi)|\,d\xi\right]^p dx\right]^{1/p} \leqq$$

$$\leqq \left[\sum_k h^{p+1}h^{p-1}\int_{x_k-h/2}^{x_k+h/2}|f''(\xi)|^p\,d\xi\right]^{1/p} = h^2\|f''\|_p.$$

For the second term we obtain

$$\left[\sum_k\int_{x_k}^{x_{k+1}}\left|\frac{1}{h}\cdot\Delta f_k-f'(x_k)\right|^p(x-x_k)^p\,dx\right]^{1/p} =$$

$$= \left[\frac{h^{p+1}}{p+1}\sum_k\left|\frac{1}{h^2}\int_{x_k-h/2}^{x_k+h/2}[f(x+h)-f(x)-f'(x_k)h]\,dx\right|^p\right]^{1/p}.$$

On the other hand, by the Taylor-formula we get

$$f(x+h) = f(x_k)+f'(x_k)(x+h-x_k)+\int_{x_k}^{x+h}(x+h-\xi)f''(\xi)\,d\xi,$$

$$f(x) = f(x_k)+f'(x_k)(x-x_k)+\int_{x_k}^{x}(x-\xi)f''(\xi)\,d\xi$$

and hence

$$f(x+h)-f(x)-f'(x_k)h = \int_{x}^{x+h}(x-\xi)f''(\xi)\,d\xi + h\int_{x_k}^{x+h}f''(\xi)\,d\xi$$

that is, continuing the above estimation

$$\left[\frac{h^{p+1}}{p+1}\sum_k\left|\frac{1}{h^2}\int_{x_k-h/2}^{x_k+h/2}\int_{x}^{x+h}(x-\xi)f''(\xi)\,d\xi\,dx+\frac{1}{h}\int_{x_k-h/2}^{x_k+h/2}\int_{x_k}^{x+h}f''(\xi)\,d\xi\,dx\right|^p\right]^{1/p} \leqq$$

$$\leqq \left[\frac{2^p h^{p+1}}{p+1}\sum_k\left\{\frac{h^{p-1}}{h^{2p}}\int_{x_k-h/2}^{x_k+h/2}\left|\int_{x}^{x+h}(x-\xi)f''(\xi)\,d\xi\right|^p dx+\right.\right.$$

$$\left.\left.+\frac{h^{p-1}}{h^p}\int_{x_k-h/2}^{x_k+h/2}\left|\int_{x_k}^{x+h}f''(\xi)\,d\xi\right|^p dx\right\}\right]^{1/p} \leqq$$

$$\leqq \left[\frac{2^p h^{p+1}}{p+1}\sum_k\left\{\frac{h^p}{h^{p+1}}\int_{x_k-h/2}^{x_k+h/2}h^{p-1}\int_{x_k-h/2}^{x_k+3h/2}|f''(\xi)|^p\,d\xi\,dx+\right.\right.$$

$$\left.\left.+\frac{1}{h}\int_{x_k-h/2}^{x_k+h/2}\left(\frac{3h}{2}\right)^{p-1}\int_{x_k-h/2}^{x_k+3h/2}|f''(\xi)|^p\,d\xi\,dx\right\}\right]^{1/p} \leqq \mathrm{const}\,h^2\|f''\|_p,$$

where we applied the inequality $|a+b|^p \leqq 2^p(|a|^p+|b|^p)$. The estimation for the sixth term in (1) is as follows:

$$\Big[\sum_k\int_{x_k}^{x_{k+1}}\Big|\int_{x_k}^{x}(x-\xi)f''(\xi)\,d\xi\Big|^p dx\Big]^{1/p} \leqq \Big[\sum_k\int_{x_k}^{x_{k+1}}\Big[\int_{x_k}^{x}(x-\xi)\,|f''(\xi)|\,d\xi\Big]^p dx\Big]^{1/p} \leqq$$

$$\leqq h\Big[\sum_k\int_{x_k}^{x_{k+1}}\Big[\int_{x_k}^{x_{k+1}}|f''(\xi)|\,d\xi\Big]^p dx\Big]^{1/p} \leqq h\Big[h\sum_k h^{p-1}\int_{x_k}^{x_{k+1}}|f''(\xi)|^p\,d\xi\Big]^{1/p} = h^2\|f''\|_p$$

and for the fifth term of (1):

$$\Big[\sum_k\int_{x_k}^{x_{k+1}}|f_k''(x-x_k)^2|^p\,dx\Big]^{1/p} = \left[\frac{h^{2p+1}}{2p+1}\sum_k\left|\frac{1}{h^2}\cdot\Delta^2 f_k\right|^p\right]^{1/p} \leqq$$

$$\leqq \frac{h^{2+1/p}}{(2p+1)^{1/p}}\,\frac{4}{h^{1/p}}\,\|f''\|_p = \mathrm{const}\,h^2\|f''\|_p.$$

Similarly

$$\Big[\sum_k\int_{x_k}^{x_{k+1}}|hf_k''(x-x_k)|^p\,dx\Big]^{1/p} \leqq \mathrm{const}\,h^2\|f''\|_p.$$

Finally, the estimation for the fourth term in (1) is very similar to that of the last terms, and we have

$$\|S_h-f\|_h \leqq \text{const}\, h^2 \|f''\|_p.$$

Now applying the lemma we get the statement.

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