

On a functional equation of sum form

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Introduction

Functional equations of sum form appear in characterizations of entropies having the sum property (see e.g. ACZÉL—DARÓCZY [1] p. 110, LOSONCZI [3]). Examples are the equations

$$(1) \quad \sum_{i=1}^k \sum_{j=1}^l f(x_i y_j) = \sum_{i=1}^k f(x_i) + \sum_{j=1}^l f(y_j)$$

and

$$(2) \quad \sum_{i=1}^k \sum_{j=1}^l f(x_i y_j) = \sum_{i=1}^k f(x_i) \sum_{j=1}^l f(y_j).$$

Here $k, l \geq 2$ are fixed integers, $(x_1, \dots, x_k) \in \Gamma_k$, $(y_1, \dots, y_l) \in \Gamma_l$ with

$$\Gamma_n = \{(x_1, \dots, x_n) \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\} \quad (n = 2, 3, \dots)$$

and $f: [0, 1] \rightarrow \mathbb{C}$ is an unknown function.

Equations (1) and (2) have been solved under various assumptions on f . If $k, l \geq 3$ and f is measurable then applying a general result of LOSONCZI ([4] Theorem 4) (1) and (2) can be reduced to the equations

$$f_1(xy) = yf_2(x) + xf_3(y) \quad x, y \in [0, 1]$$

and

$$f_1(xy) = f_2(x)f_3(y) \quad x, y \in [0, 1]$$

respectively, where

$$f_1(x) = f(x) - f(0) + xklf(0),$$

$$f_2(x) = f_1(x) - xk(l-1)f(0), \quad f_3(x) = f_1(x) - x(k-1)lf(0).$$

Thus, solving (1), (2) and similar equations is simple if f is measurable and $k, l \geq 3$. Difficulties arise however if $k=3, l=2$ ($k=2, l=3$), $k=l=2$ or if f is not measurable.

The measurable solutions and the solutions bounded on a set of positive measure of (1) are known if $k, l \geq 2$ and $k \geq 3, l \geq 2$ respectively (DARÓCZY—JÁRAI [2],

MAKSA [7]). The general solution is however not known even if $k, l \geq 3$. The only result in this direction is a representation of the solution of (1) by help of certain additive functions (LOSONCZI—MAKSA [6]).

In case of equation (2) the situation is completely different. The general solution of (2) is known if $k \geq 3, l \geq 3$ (LOSONCZI—MAKSA [5]) or $k \geq 3, l \geq 2$ and $f(x) + f(1-x) \neq 1$ (MAKSA [8]). Nothing has been known on the solutions if $k = l = 2$.

The aim of this paper is to study equation (2) with $k = l = 2$, that is the equation

$$(3) \quad \begin{aligned} f(xy) + f(x(1-y)) + f((1-x)y) + f((1-x)(1-y)) = \\ = [f(x) + f(1-x)][f(y) + f(1-y)] \quad x, y \in [0, 1]. \end{aligned}$$

We shall find all solutions of (3) in the class $C_3[0, 1]$, the space of all three times continuously differentiable complex valued functions on $[0, 1]$.

The behaviour of equation (3) is quite surprising.

First, the methods which have been used to find the general solution of (2) if $k \geq 3, l \geq 2$ completely fail.

Second, new solutions arise. Some polynomials of degree 5 are solutions of (3) but (2) with $k \geq 3, l \geq 2$ has no polynomial solution of degree 5 (different from x^5).

2. Solution of the equation (3)

Our main result is the following

Theorem. *If $f: [0, 1] \rightarrow \mathbb{C}$, $f \in C_3[0, 1]$ is a solution of (3) then*

$$(4) \quad f(x) = x^p \quad x \in [0, 1]$$

or

$$(5) \quad f(x) = qx + 1/2[1 - q \pm (1 - q)^{1/2}] \quad x \in [0, 1]$$

or

$$(6) \quad f(x) = rx^3 + 1/2[1 - 3r \pm (1 + 3r)^{1/2}]x^2 + 1/2[1 + r \mp (1 + 3r)^{1/2}]x \quad x \in [0, 1]$$

or

$$(7) \quad \begin{aligned} f(x) = \\ = sx^5 + 1/2[1 - 5s \pm (1 + 15s)^{1/2}]x^4 + 1/54[1 + 3s \mp (1 + 15s)^{1/2}](40x^3 - 15x^2 + 2x) \\ x \in [0, 1] \end{aligned}$$

where $\operatorname{Re} p \geq 3$, p, q, r, s are complex constants. Conversely, the functions (4), (5), (6), (7) are $C_3[0, 1]$ solutions of (3) if $p, q, r, s \in \mathbb{C}$ are arbitrary constants with $\operatorname{Re} p \geq 3$.

PROOF. Suppose that $f \in C_3[0, 1]$ satisfies (3). We shall deduce a differential equation for f , solve it and select the solutions of (3) out of the solutions of our differential equation. Let

$$(8) \quad h(x) = f(x) + f(1-x)$$

then

$$(9) \quad f(xy) + f(x(1-y)) + f((1-x)y) + f((1-x)(1-y)) = h(x)h(y).$$

Differentiating with respect to y ones, twice, three times we obtain

$$(10) \quad xf'(xy) - xf'(x(1-y)) + (1-x)f'((1-x)y) - (1-x)f'((1-x)(1-y)) = h(x)h'(y)$$

$$(11) \quad x^2f''(xy) + x^2f''(x(1-y)) + (1-x)^2f''((1-x)y) + (1-x)^2f''((1-x)(1-y)) = h(x)h''(y)$$

$$(12) \quad x^3f'''(xy) - x^3f'''(x(1-y)) + (1-x)^3f'''((1-x)y) - (1-x)^3f'''((1-x)(1-y)) = h(x)h'''(y).$$

Substituting $y=1$ in (9), (10), (11), (12) we get

$$(13) \quad x^n f^{(n)}(x) + (1-x)^n f^{(n)}(1-x) + (-1)^n [x^n + (1-x)^n] f^{(n)}(0) = h(x)h^{(n)}(1)$$

for $n=0, 1, 2, 3; x \in [0, 1]$.

Differentiation of (13) and multiplication by $1-x$ yields

$$(14) \quad n(1-x)^{n-1}f^{(n)}(x) + (1-x)x^n f^{(n+1)}(x) - n(1-x)^n f^{(n)}(1-x) - (1-x)^{n+1}f^{(n+1)}(1-x) + n(-1)^n [x^{n-1} - (1-x)^{n-1}] f^{(n)}(0) = (1-x)h'(x)h^{(n)}(1).$$

Adding equation (13) with n replaced by $n+1$ and equation (13) multiplied by n to equation (14) all derivatives at the point $1-x$ disappear and we have

$$(15) \quad nx^{n-1}f^{(n)}(x) + x^n f^{(n+1)}(x) + n(-1)^n x^{n-1}f^{(n)}(0) + (-1)^{n+1} [x^{n+1} + (1-x)^{n+1}] f^{(n+1)}(0) = [nh^{(n)}(1) + h^{(n+1)}(1)]h(x) + h^{(n)}(1)(1-x)h'(x).$$

For $n=0, 1, 2$ (15) gives the following system of equations:

$$(16) \quad f'(x) - f'(0) = h'(1)h(x) + h(1)(1-x)h'(x)$$

$$(17) \quad f'(x) + xf''(x) - f'(0) + [x^2 + (1-x)^2]f''(0) = [h'(1) + h''(1)]h(x) + h'(1)(1-x)h'(x)$$

$$(18) \quad 2xf''(x) + x^2f'''(x) + 2xf''(0) - [x^3 + (1-x)^3]f'''(0) = [2h''(1) + h'''(1)]h(x) + h''(1)(1-x)h'(x).$$

Denote by $L_1(x)$, $L_2(x)$, $L_3(x)$ the left hand side of (16), (17), (18) respectively. At any fixed $x \in [0, 1]$ (16), (17), (18) may be considered as a linear homogeneous system of algebraic equations for the unknowns -1 , $h(x)$, $(1-x)h'(x)$. Since this system has nontrivial solution its determinant must be zero:

$$(19) \quad \begin{vmatrix} L_1(x) & h'(1) & h(1) \\ L_2(x) & h'(1) + h''(1) & h'(1) \\ L_3(x) & 2h''(1) + h'''(1) & h''(1) \end{vmatrix} = 0.$$

Expanding this determinant by the elements of the first column we get

$$(20) \quad A_1 L_1(x) + A_2 L_2(x) + A_3 L_3(x) = 0$$

where A_i are the cofactors of L_i ($i=1, 2, 3$) in (19).

To evaluate A_3 we substitute $x=0$ in (17) We obtain that

$$f''(0) = [h'(1) + h''(1)]h(0) + h'(1)h'(0).$$

Since $h(0)=h(1)$, $h'(0)=-h'(1)$ we have

$$(21) \quad A_3 = h'(1)^2 - [h'(1) + h''(1)]h(1) = -f''(0).$$

Similarly, from (18) with $x=0$

$$(22) \quad A_2 = h(1)[2h''(1) + h'''(1)] - h'(1)h''(1) = -f'''(0).$$

To find A_1 we differentiate (17) with respect to x . The resulting equation is

$$2f''(x) + xf'''(x) + f''(0)(4x-2) = [h'(1) + h''(1)]h'(x) + h'(1)[(1-x)h''(x) - h'(x)].$$

Here the function $x \rightarrow xf'''(x)$ must be differentiable on $[0, 1]$ since all other terms in the equation are differentiable. Thus we may differentiate again and obtain

$$2f'''(x) + (xf'''(x))' + 4f''(0) = [h'(1) + h''(1)]h''(x) + h'(1)[(1-x)h'''(x) - 2h''(x)].$$

Using the relation

$$(xf'''(x))'_{x=0} = \lim_{h \rightarrow 0} hf'''(h)/h = f'''(0)$$

we get with $x=0$

$$3f'''(0) + 4f''(0) = [h'(1) + h''(1)]h''(0) + h'(1)[h'''(0) - 2h''(0)].$$

By $h''(0)=h''(1)$, $h'''(0)=-h'''(1)$ the right hand side of this equation is exactly A_1 hence

$$(23) \quad A_1 = 3f'''(0) + 4f''(0).$$

The form of our differential equation (20) thus depends on the values $f''(0)$, $f'''(0)$. We shall distinguish several cases.

Case 1. $h(1)=f(1)+f(0) \neq 1$. Then from (13) with $n=0$

$$(24) \quad h(x)[h(1)-1] = 2f(0)$$

therefore $h(x)=\text{constant}$. By (16) $f'(x)-f'(0)=0$ i.e. f is a linear function. Substitution shows that this is a solution of (3) if and only if f is of the form (5).

Case 2. $h(1)=f(1)+f(0)=1$. Then by (24) $f(0)=0$ therefore $f(1)=1$.

Subcase 2.1. $h(1)=1$, $h'(1)=0$, $f''(0)=0$. By (21) $h''(1)=0$ and from (17)

$$f'(x) + xf''(x) = f'(0).$$

The general solution of this Euler's differential equation is

$$f(x) = c_1 + c_2 \ln x + f'(0)x \quad x \in [0, 1]$$

with arbitrary constants c_1, c_2 . Since $f(0)=0, f \in C_3[0, 1]$ the constants c_1, c_2 must be zero. $f(x)=f'(0)x$ is a solution of (3) if and only if $f'(0)=0$ or $f'(0)=1$. These solutions are included in (5).

Subcase 2.2. $h(1)=1, h'(1) \neq 0, f''(0)=0$. From (21)

$$(h'(1)+h''(1))/h'(1) = h'(1)/h(1) = p \neq 0.$$

Multiplying (16) by $-p$ and adding it to (17) we get

$$(25) \quad (1-p)f'(x) + xf''(x) = (1-p)f'(0).$$

The general solution of (25) is

$$f(x) = c_1 + c_2 x^p + f'(0)x$$

with arbitrary constants c_1, c_2 . $f \in C_3[0, 1]$ implies that either $c_2=0$ or $p \in \{1, 2\} \cup \{s \in \mathbb{C} | \operatorname{Re} s \geq 3\}$. By $f(0)=0, f(1)=1$ we have $c_1=0, c_2=1-f'(0)$, i.e.

$$(26) \quad f(x) = [1-f'(0)]x^p + f'(0)x.$$

If $p=1$ then $f(x)=x$ which is a solution of form (5).

If $p \neq 1$ then the substitution of (26) in (3) yields that

$$[1-f'(0)]f'(0)[x^p+(1-x)^p-1][y^p+(1-y)^p-1] = 0.$$

Hence (26) is a solution of (3) if and only if either $1-f'(0)=0$ or $f'(0)=0$. In the first case f is of the form (5), in the second f is of the form (4) if $p \neq 2$ while f is of the form (6) (with $r=0$) if $p=2$.

Subcase 2.3. $h(1)=1, f''(0) \neq 0$. By (21), (22), (23) we get from (20) that

$$L_3(x) + uL_2(x) - (3u+4)L_1(x) = 0$$

where $u=f'''(0)/f''(0)$. Using the definition of L_i this can be written as

$$x^2 f'''(x) + (u+2)xf''(x) - 2(u+2)f'(x) + 2(u+2)f'(0) + 2f''(0)x + f'''(0)(x-x^2) = 0.$$

Integrating from 0 to x and applying the formulas

$$\int_0^x tf''(t) dt = xf'(x) - f(x) + f(0),$$

$$\int_0^x t^2 f'''(t) dt = x^2 f''(x) - 2xf'(x) + 2f(x) - 2f(0)$$

and the condition $f(0)=0$ we obtain

$$(27) \quad x^2 f''(x) + uxf'(x) - (3u+4)f(x) = \frac{x^3}{3}f'''(0) - \frac{x^2}{2}(u+2)f''(0) - 2x(u+2)f'(0).$$

Subcase 2.3.1. $h(1)=1, f''(0) \neq 0, u=f'''(0)/f''(0)=-2$. We show that this case cannot occur. The general solution of (27) is

$$f(x) = c_1 x + c_2 x^2 + \frac{f'''(0)}{6} x^3 = c_1 x + c_2 x^2 + c_3 x^3$$

with arbitrary constants c_1, c_2 . The conditions $f(1)=1, f''(0) \neq 0, u=-2$ give that $c_1+c_2+c_3=1, c_2 \neq 0, 6c_3/2c_2=-2$ that is

$$(28) \quad f(x) = c_3 x^3 - \frac{3}{2} c_3 x^2 + (1+c_3/2)x.$$

Substituting (28) in (3) we obtain, after a simple calculation, that

$$6c_3(x^2-x)(y^2-y) = 0.$$

Hence $c_3=0$ and $c_2=-\frac{3}{2}c_3=0$ which is a contradiction.

Subcase 2.3.2. $h(1)=1, f''(0) \neq 0, u=f'''(0)/f''(0) \neq -2$. In this case

$$g(x) = \frac{f'''(0)}{6} x^3 + \frac{f''(0)}{2} x^2 + f'(0)x$$

is a particular solution of (27). The homogeneous equation corresponding to (27) is an Euler's equation with characteristic equation

$$(29) \quad s^2 + (u-1)s - (3u+4) = 0.$$

We have to distinguish 2 cases again.

Subcase 2.3.2.1. $h(1)=1, f''(0) \neq 0, u \neq -2, u^2+10u+17=0$. Now $u = -5 \pm \sqrt{8}$ and the corresponding root of (29) $s_1 = \frac{u-1}{2} = -3 \pm \sqrt{2}$ has multiplicity 2. The general solution of (27) is

$$f(x) = c_1 x^{s_1} + c_2 x^{s_1} \ln x + g(x) \quad x \in (0, 1].$$

Since $f(0)=0$ and $f \in C_3[0, 1]$ c_1, c_2 must be zero. Thus

$$(30) \quad f(x) = g(x) = \frac{f'''(0)}{6} x^3 + \frac{f''(0)}{2} x^2 + f'(0)x = d_1 x^3 + d_2 x^2 + d_3 x$$

and by $f(1)=1$ $d_1+d_2+d_3=1$. Substituting (30) in (3) and using the condition $d_1+d_2+d_3=1$ we obtain

$$[(3d_1+2d_2)^2 - (9d_1+4d_2)](x^2-x)(y^2-y) = 0$$

hence (30) is a solution of (3) if and only if

$$(3d_1+2d_2)^2 = 9d_1+4d_2, \quad d_3 = 1-d_1-d_2$$

hold. Therefore, with $d_1=r$

$$d_2 = \frac{1}{2} [1-3r \pm (1+3r)^{1/2}], \quad d_3 = \frac{1}{2} [1+r \mp (1+3r)^{1/2}].$$

We obtained solution (6) (with $d_2 \neq 0, 3d_1/d_2 = -5 \pm \sqrt{8}$).

Subcase 2.3.2.2. $h(1)=1, f''(0) \neq 0, u \neq -2, u^2+10u+17 \neq 0$. In this case (29) has two distinct roots s_1, s_2 . We may suppose that $\operatorname{Re} s_1 \geq \operatorname{Re} s_2$. Due to the condition $u \neq -2$ we have $\{s_1, s_2\} \cap \{1, 2, 3\} = \emptyset$ therefore the general solution of (27) is

$$(31) \quad f(x) = c_1 x^{s_1} + c_2 x^{s_2} + g(x) \quad x \in (0, 1]$$

with arbitrary constants c_1, c_2 .

It would be a very long calculation to substitute (31) in (3). Therefore first we try to satisfy (12). Substituting (31) in (12) we obtain after some rearrangements that

$$(32) \quad e_1 k_1(x) l_1(y) + e_2 k_2(x) l_2(y) = h(x) h'''(y) \quad x, y \in (0, 1)$$

where

$$(33) \quad l_i(y) = y^{s_i-3} - (1-y)^{s_i-3} \quad y \in (0, 1)$$

$$(34) \quad k_i(x) = x^{s_i} + (1-x)^{s_i} \quad x \in (0, 1)$$

$$(35) \quad e_i = c_i s_i (s_i - 1) (s_i - 2) \quad (i = 1, 2).$$

Subcase 2.3.2.2.1. $h(1)=1, f''(0) \neq 0, u \neq -2, u^2+10u+17 \neq 0, h'''(y)=0$ if $y \in (0, 1)$. Now h is a polynomial of degree ≤ 2 and from (16) f is a polynomial of degree ≤ 3 with $f(0)=0, f(1)=1$. Thus, as in the Subcase 2.3.2.1, f must be of the form (6) (with $d_2 \neq 0, 3d_1/d_2 \neq -5 \pm \sqrt{8}$).

Subcase 2.3.2.2.2. $h(1)=1, f''(0) \neq 0, u \neq -2, u^2+10u+17 \neq 0, h'''(y_0) \neq 0$ for some $y_0 \in (0, 1), \left(y_0 \neq \frac{1}{2}\right)$. From (32) we obtain that

$$(36) \quad h(x) = B_1 k_1(x) + B_2 k_2(x)$$

with suitable constants B_1, B_2 . With (36) we get from (32) that

$$(37) \quad k_1(x)[e_1 l_1(y) - B_1 h'''(y)] + k_2(x)[e_2 l_2(y) - B_2 h'''(y)] = 0, \quad x, y \in (0, 1).$$

At this point we have to know if k_1, k_2 (and l_1, l_2) are linearly independent or not. First we deal with this question.

Lemma 1. *Let $s_1, s_2 \in \mathbb{C}, s_1 \neq s_2$. The functions k_1, k_2 defined by (34) are linearly dependent on $(0, 1)$ if and only if $s_1=0, s_2=1$ or $s_1=1, s_2=0$.*

PROOF OF LEMMA 1. $x^0 + (1-x)^0 = 2 = 2[x + (1-x)]$ thus k_1, k_2 are linearly dependent for $s_1=0, s_2=1$ and $s_1=1, s_2=0$.

We show that apart from these cases k_1, k_2 are linearly independent. Suppose that

$$(38) \quad \lambda_1 k_1(x) + \lambda_2 k_2(x) = 0 \quad x \in (0, 1)$$

holds with constants λ_1, λ_2 . Then

$$(39) \quad \lambda_1 k_1 \left(\frac{1}{2} \right) + \lambda_2 k_2 \left(\frac{1}{2} \right) = 0$$

$$(40) \quad \lambda_1 k_1'' \left(\frac{1}{2} \right) + \lambda_2 k_2'' \left(\frac{1}{2} \right) = 0$$

$$(41) \quad \lambda_1 k_1^{\text{IV}} \left(\frac{1}{2} \right) + \lambda_2 k_2^{\text{IV}} \left(\frac{1}{2} \right) = 0$$

are valid too. If $s_1 + s_2 \neq 1$ then

$$\begin{vmatrix} k_1 \left(\frac{1}{2} \right) & k_2 \left(\frac{1}{2} \right) \\ k_1'' \left(\frac{1}{2} \right) & k_2'' \left(\frac{1}{2} \right) \end{vmatrix} = 2^{4-s_1-s_2} (s_2 - s_1)(s_1 + s_2 - 1) \neq 0$$

thus by (39), (40) $\lambda_1 = \lambda_2 = 0$, k_1, k_2 are linearly independent.

If $s_1 + s_2 = 1$ but $(|s_1| + |s_2 - 1|)(|s_1 - 1| + |s_2|) \neq 0$ then

$$\begin{vmatrix} k_1'' \left(\frac{1}{2} \right) & k_2'' \left(\frac{1}{2} \right) \\ k_1^{\text{IV}} \left(\frac{1}{2} \right) & k_2^{\text{IV}} \left(\frac{1}{2} \right) \end{vmatrix} = 2^{8-s_1-s_2} s_1 s_2 (s_1 - 1)(s_2 - 1)(s_1 - s_2)(s_1 + s_2 - 5) \neq 0$$

therefore by (40), (41) $\lambda_1 = \lambda_2 = 0$ and k_1, k_2 are linearly independent.

Lemma 2. Let $s_1, s_2 \in \mathbb{C}$, $s_1 \neq s_2$. The functions l_1, l_2 defined by (33) are linearly dependent on $(0, 1)$ if and only if $s_1 = 3$ ($s_2 \neq 3$) or $s_2 = 3$ ($s_1 \neq 3$) or $s_1 = 5, s_2 = 4$ or $s_1 = 4, s_2 = 5$.

PROOF OF LEMMA 2. It is easy to see that l_1, l_2 are linearly dependent if s_1, s_2 have the values listed in Lemma 2. We show that apart from these cases l_1, l_2 are linearly independent. This follows from the fact that one of the determinants

$$\begin{vmatrix} l_1' \left(\frac{1}{2} \right) & l_2' \left(\frac{1}{2} \right) \\ l_1''' \left(\frac{1}{2} \right) & l_2''' \left(\frac{1}{2} \right) \end{vmatrix} = 2^{12-s_1-s_2} (s_1 - 3)(s_2 - 3)(s_2 - s_1)(s_1 + s_2 - 9)$$

and

$$\begin{aligned} & \begin{vmatrix} l_1''' \left(\frac{1}{2} \right) & l_2''' \left(\frac{1}{2} \right) \\ l_1^{\text{IV}} \left(\frac{1}{2} \right) & l_2^{\text{IV}} \left(\frac{1}{2} \right) \end{vmatrix} = \\ & = 2^{14-s_1-s_2} (s_1 - 3)(s_2 - 3)(s_1 - 4)(s_2 - 4)(s_1 - 5)(s_2 - 5)(s_2 - s_1)(s_1 + s_2 - 13) \end{aligned}$$

is nonzero if s_1, s_2 are not among the pairs listed in Lemma 2.

Now we return to the proof of our theorem. We remark that the assumptions on s_1, s_2 in the Subcase 2.3.2.2 and in Lemmas 1, 2 are slightly different. In Subcase 2.3.2.2 $\operatorname{Re} s_1 \equiv \operatorname{Re} s_2$, $s_1 \neq s_2$ and $\{s_1, s_2\} \cap \{1, 2, 3\} = \emptyset$ are supposed while in Lemmas 1, 2 only $s_1 \neq s_2$ is a priori assumed.

By $\{s_1, s_2\} \cap \{1, 2, 3\} = \emptyset$ and by Lemma 1 k_1, k_2 are linearly independent thus by (37)

$$(42) \quad e_1 l_1(y) = B_1 h'''(y) \quad y \in (0, 1)$$

$$(43) \quad e_2 l_2(y) = B_2 h'''(y) \quad y \in (0, 1).$$

If $e_1 = e_2 = 0$ then $c_1 = c_2 = 0$, $f(x) = g(x)$ which is a solution of (3) if and only if f is of the form (6).

If $e_1 e_2 = 0$ but one of e_1, e_2 , say e_1 , is nonzero then $e_2 = 0$, $c_2 = 0$ and

$$(44) \quad f(x) = c_1 x^{s_1} + g(x).$$

Substituting this function in (3) we get

$$(45) \quad c_1 k_1(x) k_1(y) + g(xy) + g(x(1-y)) + g((1-x)y) + g((1-x)(1-y)) = \\ = [c_1 k_1(x) + g(x) + g(1-x)][c_1 k_1(y) + g(y) + g(1-y)].$$

Differentiating (45) three times with respect to y we obtain

$$c_1 k_1(x) k_1'''(y) = [c_1 k_1(x) + g(x) + g(1-x)] c_1 k_1'''(y)$$

therefore, by

$$c_1 k_1'''(y) = c_1 s_1(s_1-1)(s_1-2)[y^{s_1-3} - (1-y)^{s_1-3}] \neq 0 \quad \text{if } y \neq \frac{1}{2}$$

we get

$$k_1(x) = c_1 k_1(x) + g(x) + g(1-x).$$

Differentiation three times shows that $c_1 = 1$ and $g(x) + g(1-x) = 0$. Hence from (45)

$$g(xy) + g(x(1-y)) + g((1-x)y) + g((1-x)(1-y)) = 0$$

which implies that

$$g(x) = 0 = \frac{f'''(0)}{6} x^3 + \frac{f''(0)}{2} x^2 + f'(0)x, \quad \text{i.e. } f''(0) = 0$$

which is a contradiction. Thus this case cannot occur.

If $e_1 e_2 \neq 0$ then $B_1 B_2 \neq 0$ since $e_1 l_1, e_2 l_2$ are nonzero functions. From (42), (43)

$$B_2 e_1 l_1(y) - B_1 e_2 l_2(y) = 0 \quad y \in (0, 1),$$

i.e. l_1, l_2 are linearly dependent on $(0, 1)$. By Lemma 2 and our condition $\operatorname{Re} s_1 \equiv \operatorname{Re} s_2$ this holds if and only if $s_1 = 5, s_2 = 4$. From (31) with

$$c_3 = \frac{f'''(0)}{6}, \quad c_4 = \frac{f''(0)}{2}, \quad c_5 = f'(0)$$

$$(46) \quad f(x) = c_1 x^5 + c_2 x^4 + c_3 x^3 + c_4 x^2 + c_5 x$$

and by $f(1)=1$

$$(47) \quad c_5 = 1 - (c_1 + c_2 + c_3 + c_4).$$

Substituting (46) in (3) we obtain after some simple calculations an equation of the form

$$(48) \quad \sum_{k=0}^3 \sum_{l=0}^3 a_{kl} p_k(x) p_l(y) = 0$$

where $p_k(x) = x^k$ ($k=0, 1, 2$) and $p_3(x) = x^4 - 2x^3$. Since p_0, p_1, p_2, p_3 are linearly independent (48) holds if and only if $a_{kl}=0$ ($k, l=0, 1, 2, 3$). A simple but long calculation gives that

$$a_{33} = 25c_1 + 4c_2 - (5c_1 + 2c_2)^2$$

$$a_{32} = a_{23} = 50c_1 + 12c_2 - (5c_1 + 2c_2)(10c_1 + 6c_2 + 3c_3 + 2c_4)$$

$$a_{31} = a_{13} = -25c_1 - 8c_2 - (5c_1 + 2c_2)(-5c_1 - 4c_2 - 3c_3 - 2c_4)$$

$$a_{30} = a_{03} = 0$$

$$a_{22} = 100c_1 + 36c_2 + 9c_3 + 4c_4 - (10c_1 + 6c_2 + 3c_3 + 2c_4)^2$$

$$a_{21} = a_{12} = -50c_1 - 24c_2 - 9c_3 - 4c_4 - (10c_1 + 6c_2 + 3c_3 + 2c_4)(-5c_1 - 4c_2 - 3c_3 - 2c_4)$$

$$a_{20} = a_{02} = 0$$

$$a_{11} = 25c_1 + 16c_2 + 9c_3 + 4c_4 - (-5c_1 - 4c_2 - 3c_3 - 2c_4)^2$$

$$a_{10} = a_{01} = a_{00} = 0.$$

It can be seen that

$$a_{31} = a_{33} - a_{32}$$

$$a_{21} = a_{32} - a_{22}$$

$$a_{11} = a_{33} + a_{22} - 2a_{32}$$

therefore $a_{kl}=0$ ($k, l=0, 1, 2, 3$) holds if and only if

$$a_{33} = 0, \quad a_{32} = 0, \quad a_{22} = 0$$

is valid. This gives the following system of equations:

$$(49) \quad 25c_1 + 4c_2 = (5c_1 + 2c_2)^2$$

$$(50) \quad 50c_1 + 12c_2 = (5c_1 + 2c_2)(10c_1 + 6c_2 + 3c_3 + 2c_4)$$

$$(51) \quad 100c_1 + 36c_2 + 9c_3 + 4c_4 = (10c_1 + 6c_2 + 3c_3 + 2c_4)^2.$$

Choosing $c_1=s$ arbitrarily we get from (49) that

$$(52) \quad c_2 = \frac{1}{2} [1 - 5s \pm (1 + 15s)^{1/2}].$$

In solving (50), (51) we may suppose that $5c_1 + 2c_2 \neq 0$ (otherwise $5c_1 + 2c_2 = 0$, $50c_1 + 12c_2 = 0$ hence $c_1 = c_2 = 0$ and then f would be a polynomial of degree ≤ 3 ,

i.e. f would be of the form (6)). From (50), (51) we get

$$3c_3 + 2c_4 = \frac{50c_1 + 12c_2}{5c_1 + 2c_2} - (10c_1 + 6c_2) = D_1$$

$$9c_3 + 4c_4 = \left(\frac{50c_1 + 12c_2}{5c_1 + 2c_2} \right)^2 - (100c_1 + 36c_2) = D_2$$

thus

$$c_3 = \frac{1}{3}(D_2 - 2D_1), \quad c_4 = \frac{1}{2}(3D_1 - D_2).$$

Expressing D_1, D_2 by help of the parameter s we obtain

$$D_1 = 5/3[1 + 3s \mp (1 + 15s)^{1/2}], \quad D_2 = 50/9[1 + 3s \mp (1 + 15s)^{1/2}]$$

further

$$(53) \quad c_3 = 20/27[1 + 3s \mp (1 + 15s)^{1/2}]$$

$$(54) \quad c_4 = -5/18[1 + 3s \mp (1 + 15s)^{1/2}]$$

and by (47)

$$(55) \quad c_5 = 1/27[1 + 3s \mp (1 + 15s)^{1/2}].$$

With $c_1 = s$, (52), (53), (54), (55) we get exactly solution (7) from (46). This concludes the proof of our theorem.

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(Received August 18, 1988)