

On the continuity of causal operators

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. There are two possible way to define the stability of an input-output system: ([1] p. 109).

a) A bounded input u produces a bounded output Tu .

b) If the bound of u is $\|u\|$ and the bound of Tu is $\|Tu\|$, then there is $c > 0$ such that $\|Tu\| < c \cdot \|u\|$. Obviously, $b) \Rightarrow a)$ but not the contrary. In a "real" system a) rather than b) can be verified however, for the operator-theoretic approach b) is needed.

So, for the application of functional analysis in the theory of linear systems, a natural problem is to find a class of linear operators when a) and b) are equivalent. Hence the following question arises:

When will an everywhere defined linear operator T be bounded in a Banach space?

FEINTUCH and SAEKS in their book [3] define a generalization of causal operator. Our main result in this paper is that for every generalized causal operator T there is a non-trivial invariant subspace $p_t \mathbf{B}$ such that T is continuous on $p_t \mathbf{B}$. Next we shall show by the method of LOY [7] that a causal and time-invariant operator in the sense of [3] p. 119, is always continuous if the shift operators are isometries. Finally, using a slight modification of HACKENBROCK's proofs [5], we prove that an operator which is passive in the sense of [3] p. 196, is also causal and continuous.

2. Let T be a linear operator on a Banach space $\mathbf{B}$ ($\mathbf{D}(T) = \mathbf{B}$ i.e. T is everywhere defined on $\mathbf{B}$); Λ be a partially ordered set such that for every $t \in \Lambda$ there is $s \in \Lambda$ so that $t < s$; $\{p^s; s \in \Lambda\}$ be bounded linear operators on $\mathbf{B}$ with the properties:

$$\text{I.} \quad \|p^s\| \leq 1 \quad \text{and} \quad p^s p^t = p^{\min(s,t)}$$

$$\text{II.} \quad p^s z = \theta \quad \text{for every } s \in \Lambda \quad \text{implies} \quad z = \theta.$$

Moreover, we define $p_s := I - p^s$ (I is the identity operator).

Definition 1. T is causal with respect to $\{p^s; s \in \Lambda\}$ if

$$(*) \quad p^s T = p^s T p^s.$$

There are several forms to express causality:

Lemma 1. *The following assertions are equivalent:*

- a) T is causal;
- b) $p_s \mathbf{B} := \{p_s x; x \in \mathbf{B}\}$ is an invariant subspace for every $s \in A$;
- c) if $p^s x = p^s y$ then $p^s T x = p^s T y$.

PROOF. a) $\Rightarrow$ b): If $p_s \mathbf{B}$ is not an invariant subspace, then there is $y \in \mathbf{B}$ such that $T p_s y \notin p_s \mathbf{B}$. On the other hand, $p_s T p_s y \in p_s \mathbf{B}$ by definition, and hence

$$T p_s \neq p_s T p_s$$

in this case.

b) $\Rightarrow$ a): If $p_s \mathbf{B}$ is an invariant subspace, i.e. $T(p_s \mathbf{B}) \subseteq p_s \mathbf{B}$, then for every $x \in \mathbf{B}$ there is $z \in \mathbf{B}$ such that

$$T p_s x = p_s z$$

and so $p_s T p_s x = p_s^2 z = p_s z$; hence

$$p_s T p_s = T p_s.$$

Substituting $p_s := I - p^s$ in this equality, (*) is obtained.

a) $\Rightarrow$ c): If $p^s x = p^s y$ and (*) is satisfied, then

$$p^s T x = p^s T p^s x = p^s T p^s y = p^s T y$$

and hence c) is obtained.

c) $\Rightarrow$ a): If $p^s x = \theta$, then it follows from c) that $p^s T x = \theta$, since T is linear. By definition

$$p^s (I - p^s) = 0$$

for every $s \in A$ and hence

$$p^s T (I - p^s) x = 0$$

for every $s \in A$ and $x \in \mathbf{B}$. Hence (*) is obtained.

Lemma 2. *For every $s \in A$, $p_s \mathbf{B}$ is a closed subset of $\mathbf{B}$.*

PROOF. If $x_n \in p_s \mathbf{B}$, then x_n has the form $p_s z_n$ and hence $p_s x_n = p_s^2 z_n = p_s z_n = x_n$. Moreover, if $x_n \rightarrow x$ then $p_s x_n \rightarrow p_s x$. We conclude

$$p_s x = x$$

and hence $x \in p_s \mathbf{B}$.

Our main result is the following

Theorem 1. *If T is causal with respect to $\{p^s; s \in A\}$, then there exist $t \in A$ and $K_t > 0$ (common for every $s \in A$!) such that*

$$\|p^s T p_t x\| \leq K_t \|p_t x\| \quad s \in A, \quad x \in \mathbf{B}.$$

This will say that each operator $p^s T$ is continuous on the invariant subspace $p_t \mathbf{B}$.

PROOF. Indirect. We suppose that for every $t \in A$ there is $s \in A$ such that $p^s T$ is unbounded on $p_t \mathbf{B}$ and we shall construct $x_0 \in \mathbf{B}$ so that $x_0 \notin \mathbf{D}(T)$.

The construction is the following. For $t_1 \in A$ there is $s_1 \in A$ such that $p^{s_1}T$ is unbounded on $p_{t_1}B$ and hence there is $p_{t_1}x_1$ so that

$$\|p_{t_1}x_1\| = 1 \quad \text{and} \quad \|p^{s_1}Tp_{t_1}x_1\| > 1.$$

For $t_2 = s_1$ there is $s_2 \in A$ such that $p^{s_2}T$ is unbounded on $p_{t_2}B$ and hence there is $p_{t_2}x_2$ so that

$$\|p_{t_2}x_2\| = 1 \quad \text{and} \quad \|p^{s_2}Tp_{t_2}x_2\| > 2^2 \left(2 + \frac{1}{2} \|Tp_{t_1}x_1\|\right)$$

moreover, $t_2 < s_2$ can be supposed since

$$\|p^s z\| = \|p^s p^{s_2} z\| \leq \|p^{s_2} z\| \quad \text{for } s < s_2.$$

In the sequel the following abbreviations will be used:

$$p^{s_k} := p^k \quad \text{resp.} \quad p_{s_k} := p_k.$$

For any integer $k > 0$ and $t_{k+1} = s_k$ there is s_{k+1} so that $p^{k+1}T$ is unbounded on $p_k B$ and hence there is $p_k x_k$ so that

$$\|p_k x_k\| = 1, \quad s_{k+1} > s_k \quad \text{and} \quad \|p^{k+1}Tp_k x_k\| > 2^k \cdot \left(k + \sum_{i=1}^{k-1} \frac{1}{2^k} \|Tp_i x_i\|\right).$$

Now, it is obvious that

$$x_0 = \sum_{k=1}^{\infty} \frac{1}{2^k} p_k x_k \in B$$

i.e. the infinite series on the right hand side is convergent.

We shall show, that for any integer $N > 0$

$$\|Tx_0\| > N$$

and hence $x_0 \notin D(T)$. In fact

$$\|Tx_0\| \cong \|p^{N+1}Tx_0\| = \sum_{k=1}^{N-1} \frac{1}{2^k} p^{N+1}Tp_k x_k + \frac{1}{2^N} p^{N+1}Tp_N x_N + p^{N+1}T \sum_{k=N+1}^{\infty} \frac{1}{2^k} p_k x_k.$$

Since T is causal, $p^{N+1}p_k = p^{N+1}(I - p^k) = 0$ for $k \geq N+1$ therefore we have

$$p^{N+1}T \sum_{k=N+1}^{\infty} \frac{1}{2^k} p_k x_k = p^{N+1}T \sum_{k=N+1}^{\infty} \frac{1}{2^k} p^{N+1}p_k x_k = \theta$$

moreover

$$\left\| \sum_{k=1}^{N-1} \frac{1}{2^k} p^{N+1}p_k x_k \right\| \leq \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_k x_k\|.$$

We conclude

$$\begin{aligned} \|Tx_0\| &\cong \left\| \frac{1}{2^N} p^{N+1}Tp_N x_N + \sum_{k=1}^{N-1} \frac{1}{2^k} p^{N+1}Tp_k x_k \right\| \cong \\ &\cong \frac{1}{2^N} \|p^{N+1}Tp_N x_N\| - \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_k x_k\| \cong \\ &\cong \left(N + \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_k x_k\| \right) - \sum_{k=1}^{N-1} \frac{1}{2^k} \|Tp_k x_k\| = N. \end{aligned}$$

From [2] II.2.7 and from the properties I and II for $\{p^s; s \in A\}$ we get the following

Corollary. *If T is causal with respect to $\{p^s; s \in A\}$ ($\mathbf{D}(T) = \mathbf{B}!$), then there exist $t \in A$ such that T is a continuous operator on the invariant subspace $p_t \mathbf{B}$.*

Remarks. I. According to Lemmas 1 and 2, if T is causal, then there are infinitely many closed invariant subspaces $p_s \mathbf{B}$ of T such that

$$p_s \mathbf{B} \subset p_t \mathbf{B}$$

for every $t \in A$. So, this is the condition which implies that there is an invariant subspace $p_t \mathbf{B} \neq \{\theta\}$ such that the restriction T to $p_t \mathbf{B}$ is continuous.

II. There is another version of the above results. Define

$$\|x\|_s := \|p_s x\| \quad s \in A; \quad x \in \mathbf{B}$$

and let $\mathbf{B}_e$ be the completion of $\mathbf{B}$ via the locally convex topology generated by the seminorms $\{\|\cdot\|_s; s \in A\}$. Then the Corollary tells us that for a causal operator T mapping $\mathbf{B}$ into $\mathbf{B}$, there is $t \in A$, such that T is bounded (and hence continuous) with respect to the seminorm $\|\cdot\|_t$ since, by the proof of Theorem 1, if this is *not* the case, then there is $x_0 \in \mathbf{B}$ such that

$$Tx_0 \notin \mathbf{B} \quad \text{but} \quad Tx_0 \in \mathbf{B}_e.$$

III. There is an obvious closed connection between the extended space $\mathbf{B}_e$ of [3] p. 173—180 and our completion $\mathbf{B}_e$. Moreover, by Definition 13 of [3] p. 179, the operator T is stable if there is $M > 0$ such that

$$\|Tx\|_s \leq M \|x\|_s \quad s \in A; \quad x \in \mathbf{B}.$$

If T is a causal operator on $\mathbf{B}$, then

$$\|Tx\|_s \leq M \|x\|_s \quad x \in \mathbf{B}$$

is automatically satisfied for *at least one* $s \in A$. (See also the Corollary of Theorem 2 in the next section.)

3. In this section, let A be a partially ordered group and for $s, t \in A$ we define

$$s > t$$

if $s - t > 0$ where 0 is the unit in A moreover

$$A_+ := \{s: s \in A, s > 0\}.$$

Definition 2. The operator U_s ($s \in A$) is called *shift-operator* if

$$p^{t+s} U_s = U_s p_t; \quad t \in A.$$

In the next, we suppose the existence of these shift operators $\{U_\tau; \tau \in A\}$ in $\mathbf{B}$.

Definition 3. The operator T is called *shift-invariant* or *time-invariant* if

$$TU_s = U_s T \quad s \in A.$$

Now we consider an *inductive limit topology* in $\mathbf{B}$. The sequence $\{x_n\}$ is called convergent and $x_n \rightarrow x$ if there exists s such that $\{x_n\} \subset p_s \mathbf{B}$ and $x \in p_s \mathbf{B}$ so that $x_n \rightarrow x$ in the seminorm $\|\cdot\|_s$.

Remark. It is easy to check that the locally convex topology generated by the seminorms $\|\cdot\|_s$; $s \in \Lambda$ is *weaker* than the inductive limit topology above considered.

Our main object in this section is to show that shift-invariant causal operators are continuous in the inductive limit topology. First we give examples for the structure described above.

Example 1. If $\mathbf{B}$ is any Banach space of time-functions, i.e. functions on the real line, p^s is the truncation operator

$$p^s f(\tau) = \begin{cases} f(\tau) & \text{if } \tau \leq s \\ 0 & \text{elsewhere,} \end{cases}$$

and U_s is the right shift by s on the real line then we obtain the causal resp. shift-invariant operators in the common sense.

Example 2. [6] Let $X=X(t)$ be a stochastic process with random variables having finite mean and variance and let $p^t X(\tau)$ be the process predicted from the part $\{X(\tau); \tau < t\}$ ("before t "). Then the transition operator T will be called causal if from $p^t X(\tau) = p^t Y(\tau)$ it follows that

$$TX(\tau) = TY(\tau) \quad \text{for } \tau < t$$

and

$$p^t TX(\tau) = p^t TY(\tau) \quad \text{for every } \tau.$$

Example 3. ([3] p. 8—9 and [6].) Let $\mathbf{H}=\mathbf{H}(R)$ be a reproducing kernel Hilbert space with kernel $R=R(\tau, t)$ ($\tau, t \in \Lambda_+$) and let p^s be the projection onto the subspace generated by

$$\{R(\cdot, t); t \leq s\}.$$

Example 4. Let $\mathbf{B}=L^1(G)$ where G is a locally compact group and $h \in L^1(G)$ such that

$$\{h * f; f \in L^1(G)\}$$

is a closed subset if $*$ is the convolution product. Let p_h be the projection onto $h * \mathbf{B} := \{h * f; f \in L^1(G)\}$ and we define $h > g$ if $h * \mathbf{B} \subset g * \mathbf{B}$. Then the operators T with the property

$$T(h * f) = h * Tf \quad h, f \in \mathbf{B}$$

are causal operators with respect to $\{p^h; h \in L^1(G)\}$.

Theorem 2. If T is causal with respect to $\{p^s; s \in \Lambda\}$, $\|U_\tau x\| = \|x\|$ for every $x \in \mathbf{B}$, $\tau \in \Lambda_+$ and

$$TU_\tau = U_\tau T \quad \tau \in \Lambda_+,$$

then for every $t \in A_+$ there is $M_t > 0$ such that

$$\|p^s T p_t x\| \leq K_t \|p_t x\| \quad s \in A_+.$$

From [2] II.2.7 and the properties I. and II. for $\{p^s; s \in A\}$ we get the following

Corollary. *A causal and shift-invariant operator T ($\mathbf{D}(T) = \mathbf{B}$!) is bounded (and hence continuous) in the inductive limit topology introduced at the beginning of this section.*

THE PROOF OF THEOREM 2. Indirect. We suppose that there exists $t \in A_+$ such that $p^s T$ is unbounded on $p_t \mathbf{B}$. Then there is a sequence $\{x_k\}$ with $\|p_t x_k\| = 1$ such that

$$\|p^s T p_t x_n\| > 2^n \left(n + \sum_{k=1}^{n-1} \frac{1}{2^k} \|T p_t x_k\| \right).$$

Now it is obvious that

$$x_0 = \sum_{k=1}^{\infty} \frac{1}{2^k} U_{(2k-1)s} p_t x_k \in \mathbf{B}$$

i.e. the infinite series on the right hand side is convergent. We shall show that for any integer $N > 0$

$$\|T x_0\| < N$$

and hence $x_0 \notin \mathbf{D}(T)$ contradicting the supposition $\mathbf{D}(T) = \mathbf{B}$. In fact

$$\begin{aligned} \|T x_0\| &\cong \|p^{2Ns} T x_0\| = \left\| \sum_{k=1}^{N-1} \frac{1}{2^k} p^{2Ns} T U_{(2k-1)s} p_t x_k + \right. \\ &\quad \left. + \frac{1}{2^N} p^{2Ns} T U_{(2N-1)s} p_t x_N + p^{2Ns} T \sum_{k=N+1}^{\infty} \frac{1}{2^k} U_{(2k-1)s} p_t x_k \right\|; \end{aligned}$$

considering the right hand side of the equality, for the first term we have

$$\left\| \sum_{k=1}^{N-1} \frac{1}{2^k} p^{2Ns} T U_{(2k-1)s} p_t x_k \right\| \leq \sum_{k=1}^{N-1} \frac{1}{2^k} \|T p_t x_k\|;$$

and for the third term

$$p^{2Ns} T \sum_{k=N+1}^{\infty} \frac{1}{2^k} U_{(2k-1)s} p_t x_k = p^{2Ns} T \sum_{k=N+1}^{\infty} \frac{1}{2^k} p^{2Ns} \cdot U_{(2k-1)s} p_t x_k = \theta$$

since T is causal, p^{2Ns} is bounded and

$$\begin{aligned} p^{2Ns} U_{(2k-1)s} p_t &= p^{2Ns} p_{t+(2k-1)s} U_{(2k-1)s}, \\ p^{2Ns} p_{t+(2k-1)s} &= p^{2Ns} (I - p^{t+(2k-1)s}) = 0 \quad \text{for } k \geq N+1 \end{aligned}$$

since $2Ns < t + (2k-1)s$ in this case.

Finally, considering the second term of the right hand side of the equality

$$\frac{1}{2^N} p^{2Ns} T U_{(2N-1)s} p_t x_N = \frac{1}{2^N} U_{(2N-1)s} p^s T p_t x_N$$

since

and hence $TU_{(2N-1)s} = U_{(2N-1)s}T$ and $p^{2Ns}U_{(2N-1)s} = U_{(2N-1)s}p^s$

$$\left\| \frac{1}{2^N} p^{2Ns} T U_{(2N-1)s} p_t x_N \right\| = \frac{1}{2^N} \|p^s T p_t x_N\|.$$

We conclude that

$$\begin{aligned} \|Tx_0\| &\cong \left\| \frac{1}{2^N} p^{2Ns} T U_{(2N-1)s} p_t x_N + \sum_{k=1}^{N-1} \frac{1}{2^k} p^{2Ns} T U_{(2k-1)s} p_t x_k \right\| \cong \\ &\cong \frac{1}{2^N} \|p^s T p_t x_N\| - \sum_{k=1}^{N-1} \frac{1}{2^k} \|T p_t x_k\| \cong \left(N + \sum_{k=1}^{N-1} \frac{1}{2^k} \|T p_t x_k\| \right) - \sum_{k=1}^{N-1} \frac{1}{2^k} \|T p_t x_k\| = N. \end{aligned}$$

4. In the spirit of [3], an operator T on a Hilbert space $\mathbf{H}$ is called *passive* if

$$\operatorname{Re}(Tx|p^t x) \geq 0 \quad x \in \mathbf{H}$$

where $\{p^t; t \in A\}$ are projection operators of $\mathbf{H}$ which satisfy the same conditions as in Section 2.

First we show that a passive operator is also causal. In fact, if

$$\mathbf{B}_T(f, g) := (Tf|p^t g) + (p^t f|Tg) \quad f, g \in \mathbf{H}$$

then $\mathbf{B}_T$ is a positive bilinear functional and hence the Cauchy inequality

$$(*) \quad |\mathbf{B}_T(f, g)|^2 \leq \mathbf{B}_T(f, f) \cdot \mathbf{B}_T(g, g)$$

is valid. So, if $p^t f = \theta$, then $\mathbf{B}_T(f, f) = 0$ and hence $\mathbf{B}_T(f, g) = 0$ for every $g \in \mathbf{H}$ and $p^t T f = \theta$ by the definition of $\mathbf{B}_T$.

We conclude that $p^t f = \theta$ implies $p^t T f = \theta$ and this is equivalent to the property c) in Lemma 1 since T is linear.

Our main result in this section is a slight generalization of [5] p. 274—275:

Theorem 3. *Every passive operator T is continuous.*

PROOF. Applied II.2.7 from [2] as in the Corollary of Theorem 1, we have only to prove that each of the operators $p^t T$ ($t \in A$) is bounded.

It follows from the definition of $\mathbf{B}_T$ that

$$(p^t T f|g) = \mathbf{B}_T(f, g) - (p^t f|Tg)$$

and

$$\mathbf{B}_T(f, f) = 2 \operatorname{Re}(p^t T f|f) \geq 0$$

moreover obviously

$$\operatorname{Re}(p^t T f|f) \leq |(p^t T f|f)|.$$

To piece together the above assertions with Cauchy inequalities, particularly with (*), we have

$$|(p^t T f|g)| \leq 2 |(p^t T f|f)|^{1/2} |(p^t T g|g)|^{1/2} + (f|f)^{1/2} \cdot (Tg|Tg)^{1/2};$$

putting

$$R(f) := |(p^t T f | f)|^{1/2} + (f | f)^{1/2}$$

$$S(g) := 2 |(p^t T g | g)|^{1/2} + (T g | T g)^{1/2}$$

we obtain by straightforward calculation

$$(**) \quad (p^t T f | g) \leq R(f) \cdot S(g).$$

Now we define

$$F_f(g) := \frac{1}{R(f)} (p^t T f | g) \quad f \neq \theta;$$

it follows from (**) that $F_f(g)$ is bounded for every $g \in \mathbf{H}$, with bound independent of f . Hence, by the uniform boundedness principle, $\{\|F_f\| : f \neq \theta\}$ is bounded i.e. there is $M > 0$ such that

$$\|F_f\| < M \quad f \neq \theta$$

and hence

$$\|p^t T f\| = \sup \{|(p^t T f | g)| : \|g\| = 1\} \leq M R(f).$$

We claim that for every operator $p^t T$ there is C such that

$$\|p^t T f\| \leq C \|f\|.$$

Indeed, from the definition of $R(f)$

$$R(f) \leq \|p^t T f\|^{1/2} \|f\|^{1/2} + \|f\|$$

hence

$$\|p^t T f\| \leq M R(f) \leq \|p^t T f\|^{1/2} M \|f\|^{1/2} + M \|f\|.$$

By completion to a full square for $\|p^t T f\|^{1/2}$ we obtain

$$\|p^t T f\| \leq \left(\frac{M}{2} + \sqrt{\frac{M^2}{4}} + M \right)^2 \|f\|.$$

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