

On completely additive functions

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Dedicated to Professor Zoltán Daróczy on his 50th birthday

1. Let $1 < q < 2$ and $L := \sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q-1}$. For every $x \in [0, L]$ let

$$(1.1) \quad \varepsilon_n(x) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} \leq x \\ 0 & \text{if } \sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} > x, \end{cases}$$

be defined by induction on n . Then we have (see [1], [2], [3])

$$(1.2) \quad x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{q^n}$$

and we call the representation (1.2) the *regular* expansion of x . By the definition (1.1), if $n \in \mathbb{N}$ such that $\varepsilon_n(x) = 0$, then

$$\sum_{i=1}^{n-1} \frac{\varepsilon_i(x)}{q^i} + \frac{1}{q^n} > x = \sum_{i=1}^{\infty} \frac{\varepsilon_i(x)}{q^i}$$

whence

$$(1.3) \quad \frac{1}{q^n} > \sum_{i=n+1}^{\infty} \frac{\varepsilon_i(x)}{q^i}.$$

Since $1 < L$, the expansion

$$(1.4) \quad 1 = \sum_{n=1}^{\infty} \frac{\varepsilon_n(1)}{q^n}$$

exists. If $\varepsilon_n(1) = 1$ for infinitely many n , then let

$$(1.5) \quad l_n := \varepsilon_n(1) \quad (n \in \mathbb{N}).$$

If $\varepsilon_n(1) = 1$ holds only for finitely many n , then let $s > 1$ be the largest index for which $\varepsilon_s(1) = 1$. Then $\varepsilon_n(1) = 0$ for every $n > s$. In this case let

$$(1.6) \quad l_n := \begin{cases} \varepsilon_i(1) & \text{if } n = k \cdot s + i \quad (1 \leq i \leq s-1, k = 0, 1, \dots) \\ 0 & \text{if } n = k \cdot s \quad (k = 1, 2, \dots). \end{cases}$$

It follows from (1.4), (1.6) that

$$(1.7) \quad 1 = \sum_{n=1}^{\infty} \frac{l_n}{q^n}$$

where $l_n=1$ for infinitely many n . We call the right-hand side of (1.7) the *infinite expansion* of the number 1.

The number $x \in [0, L]$ is called *uniquely determined* if there exists only one sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbb{N}$) for which

$$(1.8) \quad x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n}.$$

Remark. The numbers 0, L are always uniquely determined.

Let $a_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. Then we call the function

$$(1.9) \quad F(x) := \sum_{n=1}^{\infty} \varepsilon_n(x) \cdot a_n \quad (x \in [0, L])$$

additive, where $\varepsilon_n(x)$ stands for the digits 0, 1 in the regular expansion of x .

The function $F: [0, L] \rightarrow \mathbb{R}$ is called *completely additive*, if

$$(1.10) \quad F\left(\sum_{n=1}^{\infty} \frac{\varepsilon_n}{q^n}\right) = \sum_{n=1}^{\infty} \varepsilon_n \cdot F\left(\frac{1}{q^n}\right)$$

holds for every sequence $\varepsilon_n \in \{0, 1\}$ ($n \in \mathbb{N}$). By the definition of additive functions a completely additive function is additive. In our investigations we use the following result proved in [4].

Theorem 1.1. *If $F: [0, L] \rightarrow \mathbb{R}$ is a completely additive function and $F(x) \geq 0$ holds for all $x \in [0, L]$ then there exists $\alpha \in \mathbb{R}$ such that $F(x) = \alpha \cdot x$ for all $x \in [0, L]$.*

In this paper we will prove that a completely additive function is linear.

2. Let $1 < q < 2$ and $L := \sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q-1}$. For every $k > 1$, $k \in \mathbb{N}$ denote by $q^*(k)$ the unique number in the interval $(1, 2)$ for which

$$(2.1) \quad L = \sum_{n=1}^k \frac{1}{q^n}$$

holds. The following theorem has been proved in [1].

Theorem 2.1. *Let $1 < q \leq q^*(2)$ and let $F: [0, L] \rightarrow \mathbb{R}$ be a completely additive function. Then there exists an $\alpha \in \mathbb{R}$ such that $F(x) = \alpha \cdot x$ ($x \in [0, L]$).*

We will prove the following theorems.

Theorem 2.2. *Let $q = q^*(k)$ for some fixed number $k > 2$, $k \in \mathbb{N}$. Let $F: [0, L] \rightarrow \mathbb{R}$ be a completely additive function. Then there exists an $\alpha \in \mathbb{R}$ such that $F(x) = \alpha \cdot x$ for all $x \in [0, L]$.*

PROOF. Let

$$(2.2) \quad F^*(x) = F(x) - \frac{F(L) \cdot x}{L}$$

for every $x \in [0, L]$. Then F^* is a completely additive function and $F^*(0) = F^*(L) = 0$.

Let $a_n := F^*\left(\frac{1}{q^n}\right)$ ($n \in \mathbb{N}$). Then

$$(2.3) \quad \sum_{n=1}^{\infty} |a_n| < \infty, \quad \sum_{n=1}^{\infty} a_n = 0.$$

By $q = q^*(k)$

$$(2.4) \quad 1 = \sum_{i=1}^k \frac{1}{q^i}$$

consequently for every $n \in \mathbb{N}$

$$\frac{1}{q^n} = \sum_{i=1}^k \frac{1}{q^{n+i}}.$$

Since F^* is completely additive we have for every $n \in \mathbb{N}$

$$a_n = \sum_{i=1}^k a_{n+i}$$

and

$$a_{n+1} = \sum_{i=1}^k a_{n+1+i}.$$

By these

$$a_{n+1} - a_n = a_{n+k+1} - a_{n+1}$$

namely

$$(2.5) \quad a_{n+k+1} - 2 \cdot a_{n+1} + a_n = 0$$

holds for every $n \in \mathbb{N}$.

Let

$$(2.6) \quad f(z) := \sum_{i=1}^{\infty} a_i \cdot z^i.$$

Then if $|z| \leq 1$ the power series $f(z)$ is convergent, and $f(1) = f(0) = 0$. Multiply the equality (2.5) by z^{n+k+1} ($n = 1, 2, \dots$) and sum up for n . Then by (2.6) we have

$$f(z) \cdot (z^{k+1} - 2 \cdot z^k + 1) = \sum_{i=1}^{k+1} a_i \cdot z^i - 2 \cdot z^{k+1} \cdot a_1.$$

But $f(1) = f(0) = 0$ so

$$(2.7) \quad f(z) \cdot (z^{k+1} - 2 \cdot z^k + 1) = z \cdot (z-1) \cdot Q_{k-1}(z)$$

where $|z| \leq 1$, and the degree of the polynomial $Q_{k-1}(z)$ is not greater than $k-1$. Let

$$p(z) := z^{k+1} + 1, \quad g(z) := -2 \cdot z^k.$$

By $k \geq 3$

$$\infty > |g(z)| > |p(z)| > 0$$

if $|z|=1$. Hence, by the theorem of Rouché the functions $p+g, g$ have the same number of zeros on the disk $|z|<1$. But if $1<|z|<1$ then

$$|p(z)+g(z)| > 0.$$

As the function g has k roots on the closed disc $|z|\leq 1$, the function $p+g$ has exactly k roots on it. Thus, by (2.6) the polynomial $Q_{k-1}(z)$ has k roots, but in this case $Q_{k-1}(z)\equiv 0$. That is $f(z)\equiv 0$, ($|z|\leq 1$) hence

$$(2.8) \quad a_n = 0$$

for every $n\in\mathbf{N}$. So we have

$$(2.9) \quad F\left(\frac{1}{q^n}\right) = \frac{F(L)}{L} \cdot \frac{1}{q^n}$$

for every $n\in\mathbf{N}$. That is by $c:=\frac{F(L)}{L}$

$$(2.10) \quad F(x) = c \cdot x$$

for all $x\in[0, L]$. $\square$

Theorem 2.3. Let $q^*(2)<q<2$ be a fixed number for which

$$(2.11) \quad 1 \neq \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^n}$$

holds for every $n>2$, $n\in\mathbf{N}$. Moreover let $F: [0, L]\rightarrow\mathbf{R}$ $\left(L:=\sum_{n=1}^{\infty}\frac{1}{q^n}=\frac{1}{q-1}\right)$ be a completely additive function defined by the sequence $a_n\in\mathbf{R}$, $\left(\sum_{n=1}^{\infty}|a_n|<\infty\right)$. Then F does not change sign.

PROOF. Indirect. Suppose that F changes sign, and let

$$(2.12) \quad P:=\{n|n\in\mathbf{N}, a_n>0\}.$$

Then $P\neq\mathbf{N}$, $P\neq\emptyset$. Let

$$(2.13) \quad x:=\sum_{n\in P}\frac{1}{q^n}$$

$$(2.14) \quad y:=\sum_{m\notin P}\frac{1}{q^m}.$$

So $x, y\in(0, L)$ and these numbers are uniquely determined (see in [1]). Thus the regular expansions of x, y are just the right-hand sides of (2.13), (2.14). Moreover the sets $\mathbf{N}\setminus P, P$ contain infinitely many numbers.

Denote by $n_0\in\mathbf{N}$ that unique number for which

$$(2.15) \quad \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^{n_0-1}} < 1 < \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^{n_0}}$$

holds, then by $q^*(2) < q$, and (2.11) this number exists and $n_0 \geq 3$. Hence we have

$$(2.16) \quad 1 = \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^{n_0-1}} + \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^i}$$

where l_i denotes the digits 0, 1 in the infinite expansion of the number 1.

Let

$$(2.17) \quad t := \min \{n | n \in \mathbb{N}, n > n_0, n \notin P, n+1 \in P\}.$$

Then by (1.3), (2.13), (2.14), we have

$$(2.18) \quad \sum_{n>t, n \in P} \frac{1}{q^n} < \frac{1}{q^t}$$

$$(2.19) \quad \sum_{m>t+1, m \notin P} \frac{1}{q^m} < \frac{1}{q^{t+1}}.$$

We distinguish the following cases:

$$(2.20) \quad \text{I.} \quad \sum_{n>t, n \in P} \frac{1}{q^n} = \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}}$$

$$(2.21) \quad \text{II.} \quad \sum_{n>t, n \in P} \frac{1}{q^n} < \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}}$$

$$(2.22) \quad \text{III.} \quad \sum_{n>t, n \in P} \frac{1}{q^n} > \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}}.$$

It is easy to see that there are no other cases.

In the case I. by (2.16), (2.20) we have

$$(2.23) \quad \frac{1}{q^{t-n_0}} = \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \sum_{n>t, n \in P} \frac{1}{q^n}.$$

Hence, since F is completely additive

$$F\left(\frac{1}{q^{t-n_0}}\right) = a_{t-n_0}$$

$$F\left(\frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \sum_{n>t, n \in P} \frac{1}{q^n}\right) = a_{t-n_0+1} + \dots + a_{t-1} + \sum_{n>t, n \in P} a_n$$

that is

$$a_{t-n_0} = a_{t-n_0+1} + \dots + a_{t-1} + \sum_{n>t, n \in P} a_n$$

and by the definition of the set P we have

$$(2.24) \quad a_{t-n_0} > a_{t-n_0+1} + \dots + a_{t-1}.$$

Now we consider the following number

$$(2.25) \quad x := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{m>t+1, m \notin P} \frac{1}{q^m}.$$

Then $x \in [0, L]$ and by (2.15)

$$(2.26) \quad \frac{1}{q^{t-n_0}} < x.$$

That is, by (1.3) the regular expansion of x , $(\varepsilon_n(x))$ is different from the right-hand side of (2.25). Moreover, by $t+1 \in P$, (2.19), (2.20), (2.16) we have

$$x < \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{n>t, n \in P} \frac{1}{q^n} = \frac{1}{q^{t-n_0}} + \frac{1}{q^t},$$

whence

$$(2.27) \quad x = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(x)}{q^i}.$$

By the definition of the set P , and since F is completely additive we have

$$\begin{aligned} F(x) &= a_{t-n_0} + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i = a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{m>t+1, m \notin P} a_m \equiv \\ &\equiv a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i. \end{aligned}$$

That is

$$a_{t-n_0} \equiv a_{t-n_0+1} + \dots + a_{t-1} + a_t$$

and, by $t \notin P$ $a_t \equiv 0$, so we have

$$(2.28) \quad a_{t-n_0} \equiv a_{t-n_0+1} + \dots + a_{t-1}$$

which contradicts (2.24).

In the case II. we first consider the following number

$$(2.29) \quad x := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{n>t, n \in P} \frac{1}{q^n}.$$

By (2.15)

$$(2.30) \quad \frac{1}{q^{t-n_0}} < x,$$

that is, by (1.3) the regular expansion of x , $(\varepsilon_n(x))$ is different from the right-hand side of (2.29). Moreover, by (2.21), (2.16) we have

$$x < \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}} = \frac{1}{q^{t-n_0}} + \frac{1}{q^t}.$$

Consequently

$$(2.31) \quad x = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(x)}{q^i}.$$

Hence we have

$$\begin{aligned}
 a_{t-n_0} + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i &= a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{n>t, n \in P} a_n \cong \\
 &\cong a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i, \\
 (2.32) \quad a_{t-n_0} &\cong a_{t-n_0+1} + \dots + a_{t-1} + a_t.
 \end{aligned}$$

Now we consider the following number

$$(2.33) \quad y := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{m>t+1, m \notin P} \frac{1}{q^m}.$$

By (2.15)

$$(2.34) \quad \frac{1}{q^{t-n_0}} < y,$$

that is, by (1.3) the regular expansion of y , $(\varepsilon_n(y))$ is different from the right-hand side of (2.33). Moreover, by $t+1 \in P$, (2.19), (2.21), (2.16) we have

$$\begin{aligned}
 y &< \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{n>t, n \in P} \frac{1}{q^n} < \\
 &< \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{i=n_0+1}^{\infty} \frac{l_i}{q^{t-n_0+i}} = \frac{1}{q^{t-n_0}} + \frac{1}{q^t},
 \end{aligned}$$

and by (2.15) we have

$$(2.35) \quad y = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(y)}{q^i} > \frac{1}{q^{t-n_0}} + \sum_{m>t+1, m \notin P} \frac{1}{q^m}.$$

Hence by the definition (1.1) there exist $k \in \mathbb{N}$ such that $\varepsilon_{t+k}(y) = 1$ and $t+k \in P$. So we have

$$\begin{aligned}
 a_{t-n_0} + \sum_{i=t+1}^{\infty} \varepsilon_i(y) \cdot a_i &= a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{m>t+1, m \notin P} a_m < \\
 &< a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{i=t+1}^{\infty} \varepsilon_i(y) \cdot a_i, \\
 (2.36) \quad a_{t-n_0} &< a_{t-n_0+1} + \dots + a_{t-1} + a_t
 \end{aligned}$$

which contradicts (2.32).

In the case III. we first consider the following number

$$(2.37) \quad x := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \sum_{n>t, n \in P} \frac{1}{q^n}.$$

By (2.16), (2.22)

$$(2.38) \quad \frac{1}{q^{t-n_0}} < x,$$

that is, by (1.3) the regular expansion of x , $(\varepsilon_n(x))$ is different from the right-hand side of (2.37). Furthermore, by (2.15), (2.18)

$$x < \frac{1}{q^{t-n_0}} + \sum_{n>t, n \in P} \frac{1}{q^n} < \frac{1}{q^{t-n_0}} + \frac{1}{q^t}.$$

Consequently

$$(2.39) \quad x = \frac{1}{q^{t-n_0}} + \sum_{i=t+1}^{\infty} \frac{\varepsilon_i(x)}{q^i}$$

and there exists $k \in \mathbb{N}$ such that $\varepsilon_{t+k}(x) = 0$ and $t+k \in P$, whence

$$(2.40) \quad \begin{aligned} a_{t-n_0} + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i &= a_{t-n_0+1} + \dots + a_{t-1} + \sum_{n>t, n \in P} a_n > \\ &> a_{t-n_0+1} + \dots + a_{t-1} + \sum_{i=t+1}^{\infty} \varepsilon_i(x) \cdot a_i. \end{aligned}$$

Now consider the following number

$$(2.41) \quad y := \frac{1}{q^{t-n_0+1}} + \dots + \frac{1}{q^{t-1}} + \frac{1}{q^t} + \sum_{m>t+1, m \notin P} \frac{1}{q^m}.$$

By (2.15)

$$(2.42) \quad \frac{1}{q^{t-n_0}} < y,$$

that is, by (1.3) the regular expansion of y , $(\varepsilon_n(y))$ is different from the right-hand side of (2.41). Moreover, by (2.15), (2.19) and $n_0 \geq 3$ we have

$$y < \frac{1}{q^{t-n_0}} + \frac{1}{q^t} + \sum_{m>t+1, m \notin P} \frac{1}{q^m} < \frac{1}{q^{t-n_0}} + \frac{1}{q^t} + \frac{1}{q^{t+1}} < \frac{1}{q^{t-n_0}} + \frac{1}{q^{t-1}}.$$

Thus

$$(2.43) \quad y = \frac{1}{q^{t-n_0}} + \sum_{i=t}^{\infty} \frac{\varepsilon_i(y)}{q^i}$$

and so we have

$$\begin{aligned} a_{t-n_0} + \sum_{i=t}^{\infty} \varepsilon_i(y) \cdot a_i &= a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{m>t+1, m \notin P} a_m \leq \\ &\leq a_{t-n_0+1} + \dots + a_{t-1} + a_t + \sum_{i=t+1}^{\infty} \varepsilon_i(y) \cdot a_i, \\ a_{t-n_0} &< a_{t-n_0+1} + \dots + a_{t-1} + a_t \cdot (1 - \varepsilon_t(y)) \end{aligned}$$

but $t \notin P$ because $a_t \equiv 0$ hence we have

$$(2.44) \quad a_{t-n_0} \leq a_{t-n_0+1} + \dots + a_{t-1}$$

which contradicts (2.40).

So we have a contradiction in each case. Our proof is complete. $\square$

Corollary. *By Theorem 1.1 in this case the completely additive function is linear.*

By Theorem 2.1, Theorem 2.2 and the Corollary we have

Theorem 2.3. *Let $1 < q < 2$, $L := \sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q-1}$ and let $F: [0, L] \rightarrow \mathbf{R}$ be a completely additive function. Then there exists an $\alpha \in \mathbf{R}$ such that $F(x) = \alpha \cdot x$ for all $x \in [0, L]$.*

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(Received December 15, 1987)