

A theorem of Sjogren and Hartley on dimension subgroups

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In this note we give a direct proof of a generalized version of a result due to SJOGREN [5] and HARTLEY [4], and point out a far reaching application. Let $H = H_1 \cong H_2 \cong \dots$ and $K = K_1 \cong K_2 \cong \dots$ be series of normal subgroups of a group F and let $\{D_{k,l}; 1 \leq k < l\}$ be a family of normal subgroups of F such that (a) $D_{k,k+1} = H_k \cdot K_{k+1}$, (b) $H_k \cdot K_l \leq D_{k,l}$ and (c) $D_{k,l+1} \leq D_{k,l}$ for all $k < l$. We prove the following theorem whose foundation lies in [5] and [4].

Theorem A. *For each $2 \leq k+m \leq n+1, k, m, n \geq 1$, let there exist a positive integer $a(k)$, depending only on k and n , such that $(K_{k+m} \cap D_{k,k+m+1})^{a(k)} \leq D_{k+1,k+m+1} \cdot H_k$. Then $(D_{1,n+2})^{a(1,n+1)} \leq H_1 \cdot K_{n+2}$, where $a(1,n+1) = a(1) \binom{n}{1} \dots a(n) \binom{n}{n}$.*

PROOF. We prove by induction on $1 \leq m \leq n$, that $(D_{k,k+m+1})^{a(k,m+1)} \leq H_k \cdot K_{k+m+1}$, where $a(k,m+1) = a(k) \binom{m}{1} \dots a(k+m-1) \binom{m}{m}$; the proof of the theorem then follows with $k=1, m=n$. When $m=1$, we have $D_{k,k+2} \leq D_{k,k+1} = H_k \cdot K_{k+1}$ and it follows that $D_{k,k+2} \leq H_k \cdot K_{k+1} \cap D_{k,k+2} \leq (K_{k+1} \cap D_{k,k+2}) \cdot H_k$, since $H_k \leq D_{k,k+2}$. Thus by the given hypothesis $(D_{k,k+2})^{a(k)} \leq D_{k+1,k+2} \cdot H_k = H_{k+1} \cdot K_{k+2} \cdot H_k = H_k \cdot K_{k+2}$ as required since $a(k) = a(k, 2)$. For the inductive step, let $m \geq 2$ and assume the result for $m-1$. Then we have $D_{k,k+m+1} \leq D_{k,k+(m-1)+1}$ and the induction hypothesis yields $(D_{k,k+m+1})^{a(k,m)} \leq H_k \cdot K_{k+m} \cap D_{k,k+m+1} \leq (K_{k+m} \cap D_{k,k+m+1}) \cdot H_k$, since $H_k \leq D_{k,k+m+1}$. By the given hypothesis it now follows that $(D_{k,k+m+1})^{a(k,m) \cdot a(k)} \leq (K_{k+m} \cap D_{k,k+m+1})^{a(k)} \cdot H_k \leq D_{k+1,k+m+1} \cdot H_k$. On the other hand, $D_{k+1,k+m+1} = D_{k+1,k+1+m}$ gives, by the induction hypothesis,

$$(D_{k+1,k+m+1})^{a(k+1,m)} \leq H_{k+1} \cdot H_{k+m+1}.$$

This yields, in turn,

$$(D_{k,k+m+1})^{a(k,m) \cdot a(k) \cdot a(k+1,m)} \leq H_k \cdot K_{k+m+1}.$$

It only remains to verify that $a(k,m+1) = a(k,m) \cdot a(k) \cdot a(k+1,m)$. Indeed, we have

$$a(k,m) = a(k) \binom{m-1}{1} \dots a(k+m-2) \binom{m-1}{m-1}$$

and

$$a(k+1,m) = a(k+1) \binom{m-1}{1} \dots a(k+1+m-2) \binom{m-1}{m-1}.$$

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Thus

$$\begin{aligned}
 & a(k, m) \cdot a(k) \cdot a(k+1, m) = \\
 & = a(k)^{\binom{m-1}{0} + \binom{m-1}{1}} a(k+1)^{\binom{m-1}{1} + \binom{m-1}{2}} \dots a(k+m-2)^{\binom{m-1}{m-2} + \binom{m-1}{m-1}} a(k+m-1) = \\
 & = a(k)^{\binom{m}{1}} a(k+1)^{\binom{m}{2}} \dots a(k+m-2)^{\binom{m}{m-1}} a(k+m-1)^{\binom{m}{m}} = \\
 & = a(k, m+1).
 \end{aligned}$$

An application. Let $G = F/RF''$ be a finitely generated metabelian group with F/RF' finite. Let $\mathbf{f} = \mathbf{Z}F(F-1)$ be the augmentation ideal of the free group ring $\mathbf{Z}F$. Define $\mathbf{r}(1) = \mathbf{r} = \mathbf{Z}F(R-1)$ and for $k \geq 2$, $\mathbf{r}(k) = \sum_{i+j=k-1} \mathbf{f}^i \mathbf{r} \mathbf{f}^j$, $i, j \geq 0$. Also, define $R(1) = R$, $R(k) = [R(k-1), F]$, $k \geq 2$. Let $\mathbf{a} = \mathbf{Z}F(F'-1)$ and set $D(k, l) = F \cap (1 + \mathbf{r}(k) + \mathbf{a}^2 + \mathbf{f}^l)$ for all $1 \leq k < l$. It is easy to prove that $D(k, k+1) = R(k) \cdot F'' \cdot \gamma_{k+1}(F)$. Thus, with $H_k = R(k) \cdot F''$, $K_l = \gamma_l(F)$ and $D_{k,l} = D(k, l)$, the family $\{D(k, l); 1 \leq k < l\}$, of normal subgroups of F , satisfies the conditions (a), (b) and (c) stated earlier. Let G be a finitely generated metabelian p -group and let $n \leq 2p-3$. Then, for a suitable free presentation $1 \rightarrow RF'' \rightarrow F \rightarrow G \rightarrow 1$ of G , by a delicate blend of techniques from CLIFF—HARTLEY [1] and GUPTA [2] it is possible to choose $a(k) = k!$ for $k = 1, \dots, p-1$ and $a(k) = (n-k+2)!$ for $k = p, \dots, n+1$ such that $(\gamma_{k+m}(F) \cap D(k, k+m+1))^{a(k)} \leq D(k+1, k+m+1) \cdot R(K) \cdot F''$. It follows by Theorem A that $(D(1, n+2))^{a(1, n+1)} \leq R \cdot F'' \cdot \gamma_{n+2}(F)$, where $a(1, n+1) = a(1)^{\binom{n}{1}} \dots a(n)^{\binom{n}{n}}$ is coprime to p . Thus $F \cap (1 + \mathbf{r} + \mathbf{a}^2 + \mathbf{f}^{n+2}) \leq R \cdot F'' \cdot \gamma_{n+2}(F)$ and consequently we have the following important result,

Theorem B. *Let G be a finite metabelian p -group. Then for $n \leq 2p-1$ the n -th dimension subgroup $D_n(G)$ coincides with the n -th lower central subgroup $\gamma_n(G)$.*

[Details will be published elsewhere. The earlier best-known results in this direction are: $D_n(G) = \gamma_n(G)$ for $n \leq p+2$, p odd (GUPTA—TAHARA [3]), and for arbitrary p -groups $D_n(G) = \gamma_n(G)$ for $n \leq p+1$ (SJOGREN [5]).]

References

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CANADA

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