

# On the $n$ -dimensional $SE$ -connection and its conformal change\*

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*Abstract.* The purpose of the present paper is to investigate properties of the  $n$ -dimensional  $SE$ -connection and its change induced by the conformal change (5.2). In the present paper, we introduce the concept of the  $SE$ -connection, find a necessary and sufficient condition for the existence of a unique  $SE$ -connection, and derive its representations. We also derive a useful representation of its change induced by the conformal change.

## I. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ (1957). CHUNG (1968, 1974) also investigated the same topic in 4-dimensional  $*g$ -unified field theory. In the present paper, we first introduce the concept of an  $n$ -dimensional  $SE$ -manifold,  $SEX_n$ , a generalized Riemannian space connected by the  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  which is both semi-symmetric and Einstein. In the sequel, we find a necessary and sufficient condition for the existence of a unique  $SE$ -connection and derive several representations of the unique  $SE$ -connection. In the last section we finally investigate the change  $\Gamma_{\lambda\mu}^\nu \rightarrow \bar{\Gamma}_{\lambda\mu}^\nu$  of two  $n$ -dimensional  $SE$ -connections induced by the conformal change (5.2).

All considerations and results presented in the present paper hold for all classes, all possible indices of inertia, and an arbitrary  $n > 1$ .

## II. Preliminaries

This section is a brief collection of definitions, notations, and basic results which are needed in our subsequent considerations. The detailed proofs are given in CHUNG (1963, 1981, 1982), HLAVATÝ (1957), and MISHRA (1958).

Let  $X_n$  be a generalized  $n$ -dimensional Riemannian space referred to a real coordinate system  $x^\nu$ , which obeys only coordinate transformations  $x^\nu \rightarrow x^{\nu'}$  for which

$$(2.1) \quad \text{Det} \left( \left( \frac{\partial x^{\nu'}}{\partial x^\nu} \right) \right) \neq 0.$$

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The space  $X_n$  is endowed with a general real nonsymmetric tensor  $g_{\lambda\mu}$  which may be split into its symmetric part  $h_{\lambda\mu}$  and skew-symmetric part  $k_{\lambda\mu}^{(*)}$ :

$$(2.2) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu},$$

where

$$(2.3) \quad g \stackrel{\text{df}}{=} \text{Det}((g_{\lambda\mu})) \neq 0, \quad h \stackrel{\text{df}}{=} \text{Det}((h_{\lambda\mu})) \neq 0, \quad k \stackrel{\text{df}}{=} \text{Det}((k_{\lambda\mu})).$$

We may define a unique tensor  $h^{\lambda\nu}$  by

$$(2.4) \quad h_{\lambda\mu} h^{\lambda\nu} \stackrel{\text{df}}{=} \delta_{\mu}^{\nu},$$

which together with  $h_{\lambda\mu}$  will serve for raising and/or lowering indices of tensors in  $X_n$  in the usual manner.

The space  $X_n$  is connected by a general real connection  $\Gamma_{\lambda\mu}^{\nu}$  with the following transformation rule:

$$(2.5) \quad \Gamma_{\lambda'\mu'}^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^{\alpha}} \left( \frac{\partial x^{\beta}}{\partial x^{\lambda'}} \frac{\partial x^{\gamma}}{\partial x^{\mu'}} \Gamma_{\beta\gamma}^{\alpha} + \frac{\partial^2 x^{\alpha}}{\partial x^{\lambda'} \partial x^{\mu'}} \right).$$

It may also be decomposed into its symmetric part  $A_{\lambda\mu}^{\nu}$  and its skew-symmetric part  $S_{\lambda\mu}^{\nu}$ , called the torsion tensor of  $\Gamma_{\lambda\mu}^{\nu}$ :

$$(2.6) \quad \Gamma_{\lambda\mu}^{\nu} = A_{\lambda\mu}^{\nu} + S_{\lambda\mu}^{\nu}; \quad A_{\lambda\mu}^{\nu} = \Gamma_{(\lambda\mu)}^{\nu}, \quad S_{\lambda\mu}^{\nu} = \Gamma_{[\lambda\mu]}^{\nu}.$$

A connection  $\Gamma_{\lambda\mu}^{\nu}$  is said to be *Einstein* if it satisfies the following Einstein equations:

$$(2.7)a \quad \partial_{\omega} g_{\lambda\mu} - \Gamma_{\lambda\omega}^{\alpha} g_{\alpha\mu} - \Gamma_{\omega\mu}^{\alpha} g_{\lambda\alpha} = 0,$$

or equivalently

$$(2.7)b \quad D_{\omega} g_{\lambda\mu} = 2S_{\omega\mu}^{\alpha} g_{\lambda\alpha}$$

where  $D_{\omega}$  is the symbolic vector of the covariant derivative with respect to  $\Gamma_{\lambda\mu}^{\nu}$ . The space  $X_n$  in this case is a generalization of the space-time  $X_4$ . A connection  $\Gamma_{\lambda\mu}^{\nu}$  is said to be *semi-symmetric* if its torsion tensor  $S_{\lambda\mu}^{\nu}$  is of the form

$$(2.8) \quad S_{\lambda\mu}^{\nu} = 2\delta_{[\lambda}^{\nu} X_{\mu]}$$

for an arbitrary vector  $X_{\mu}$ . A connection  $\Gamma_{\lambda\mu}^{\nu}$  which is both semi-symmetric and Einstein is called an *SE-connection*. An  $n$ -dimensional *SE-manifold*, denoted by  $SEX_n$  in our further considerations, is a space  $X_n$  on which the differential-geometric structure is imposed by  $g_{\lambda\mu}$  through an *SE-connection*  $\Gamma_{\lambda\mu}^{\nu}$ .

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\* Throughout the present paper, all Greek indices take the values  $1, 2, \dots, n$  and follow the summation convention.

The following quantities are frequently used in the present paper:

$$(2.9)a \quad g \stackrel{\text{df}}{=} \frac{g}{h}, \quad k \stackrel{\text{df}}{=} \frac{k}{h};$$

$$(2.9)b \quad \sigma \stackrel{\text{df}}{=} \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases};$$

$$(2.9)c \quad {}^{(0)}k_\lambda^\nu \stackrel{\text{df}}{=} \delta_\lambda^\nu, \quad {}^{(p)}k_\lambda^\nu \stackrel{\text{df}}{=} {}^{(p-1)}k_\lambda^\alpha k_\alpha^\nu, \quad (p = 1, 2, \dots);$$

$$(2.9)d \quad K_p \stackrel{\text{df}}{=} k_{[\alpha_1}^{\alpha_1} k_{\alpha_2}^{\alpha_2} \dots k_{\alpha_p}^{\alpha_p]}, \quad (p = 0, 1, 2, \dots);$$

$$(2.9)e \quad T_{\omega\mu\nu}^{pqr} \stackrel{\text{df}}{=} {}^{(p)}k_\omega^\alpha {}^{(q)}k_\mu^\beta {}^{(r)}k_\nu^\gamma T_{\alpha\beta\gamma};$$

$$(2.9)f \quad K_{\omega\mu\nu} \stackrel{\text{df}}{=} \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} + \nabla_\nu k_{\omega\mu}.$$

Here  $T_{\omega\mu\nu}$  is an arbitrary tensor and  $\nabla_\omega$  is the symbolic vector of the covariant derivative with respect to the Christoffel symbols  $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$  defined by  $h_{\lambda\mu}$ .

In a general  $X_n$  it may be easily shown that

$$(2.10)a \quad {}^{(p)}k_{\lambda\mu} \quad \text{and} \quad {}^{(p)}k^{\lambda\nu} \quad \text{are} \begin{cases} \text{symmetric if } p \text{ is even} \\ \text{skew-symmetric if } p \text{ is odd}; \end{cases}$$

$$(2.10)b \quad K = 1, \quad K_n = k \quad \text{if } n \text{ is even}, \quad K_p = 0 \quad \text{if } p \text{ is odd};$$

$$(2.10)c \quad T_{\omega\mu\nu}^{000} = T_{\omega\mu\nu}, \quad T_{\omega\mu\nu}^{pqr} = -T_{\mu\omega\nu}^{qpr} \quad \text{if } T_{\omega\mu\nu} = -T_{\mu\omega\nu};$$

$$(2.10)d \quad g = \sum_{s=0}^{n-\sigma} K_s;$$

$$(2.10)e \quad \sum_{s=0}^{n-\sigma} K_s {}^{(n-s)}k_\lambda^\nu = 0 \quad (\text{Recurrence relation}).$$

Here and in what follows we assume that  $s$  takes only even integers in the given range.

If the system (2.7) admits a solution  $\Gamma_{\lambda\mu}^\nu$ , it must be of the form

$$(2.11) \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + U_{\lambda\mu}^\nu + S_{\lambda\mu}^\nu,$$

where

$$(2.12) \quad U_{\lambda\mu}^\nu = 2 S_{(\lambda\mu)}^{\nu 001}.$$

### III. The SE-connection $\Gamma_{\lambda\mu}^v$

In this section we shall obtain a necessary and sufficient condition for a general connection  $\Gamma_{\lambda\mu}^v$  of  $X_n$  to be a unique SE-connection and derive some useful representations of the SE-connection.

**Theorem (3.1).** *If there is an SE-connection  $\Gamma_{\lambda\mu}^v$ , it must be of the form*

$$(3.1) \quad \Gamma_{\lambda\mu}^v = \left\{ \begin{smallmatrix} v \\ \lambda\mu \end{smallmatrix} \right\} + 2k_{(\lambda}^v X_{\mu)} + 2\delta_{[\lambda}^v X_{\nu]}$$

for a vector  $X_\mu$ .

PROOF. If  $\Gamma_{\lambda\mu}^v$  is an SE-connection, it must be of the form (2.11). Substituting (2.8) and

$$(3.2) \quad U_{\lambda\mu}^v = 2h^{v\alpha}(\delta_{[\alpha}^\beta X_{\lambda]} k_{\mu\beta} - \delta_{[\alpha}^\beta X_{\mu]} k_{\lambda\beta}) = 2k_{(\lambda}^v X_{\mu)}$$

into (2.11), we have (3.1).

In the next Theorem we shall assume that the symmetric tensor

$$(3.3) \quad P_{\lambda\mu} \stackrel{\text{df}}{=} {}^{(2)}k_{\lambda\mu} - h_{\lambda\mu}$$

is of rank  $n^{(*)}$ , so that there exists a unique symmetric tensor  $Q^{\lambda\nu} = Q^{\nu\lambda}$  satisfying

$$(3.4) \quad P_{\lambda\mu} Q^{\lambda\mu} \stackrel{\text{df}}{=} \delta_\mu^v.$$

**Theorem (3.2).** *There exists a unique SE-connection  $\Gamma_{\lambda\mu}^v$  if, and only if there is a vector  $X_\mu$  such that*

$$(3.5) \quad \nabla_\omega k_{\lambda\mu} + 2P_{\omega[\lambda} X_{\mu]} = 0.$$

The vector  $X_\mu$  satisfying (3.5) is unique and may be given by

$$(3.6) \quad X_\mu = \frac{1}{1-n} Q^{\alpha\beta} \nabla_\alpha k_{\beta\mu}.$$

PROOF. Multiply both sides of (3.5) by  $Q^{\omega\nu}$  and contract for  $\nu$  and  $\lambda$  in order to obtain (3.6). The uniqueness of the vector  $X_\mu$  satisfying (3.5) is obvious.

Suppose that  $\Gamma_{\lambda\mu}^v$  is an SE-connection. Then it must be of the form (3.1) and satisfy (2.7). Substituting (3.1) into (2.7)a, we have the condition (3.5). Conversely, assume now that there exists a vector  $X_\mu$  satisfying (3.5). With this vector  $X_\mu$ , define a semi-symmetric connection  $\Gamma_{\lambda\mu}^v$  by (3.1). This connection  $\Gamma_{\lambda\mu}^v$  is clearly Einstein since it satisfies (2.7)a in virtue of our assumption (3.5).

Besides the SE-connection  $\Gamma_{\lambda\mu}^v$  defined by (3.1) and (3.5), assume that there exists another SE-connection

$$(3.7) \quad {}^* \Gamma_{\lambda\mu}^v = \left\{ \begin{smallmatrix} v \\ \lambda\mu \end{smallmatrix} \right\} + 2k_{(\lambda}^v {}^* X_{\mu)} + 2\delta_{[\lambda}^v {}^* X_{\nu]}, \quad {}^* X_\mu \neq X_\mu.$$

\* In Remark (4.3) we shall show that the rank of  $P_{\lambda\mu}$  is  $n$ .

Then, in virtue of the above discussion,  ${}^*X_\mu$  must satisfy

$$(3.8) \quad \nabla_\omega k_{\lambda\mu} + 2P_{\omega[\lambda} {}^*X_{\mu]} = 0.$$

Applying the same method used to derive (3.6), we have from (3.8)

$${}^*X_\mu = \frac{1}{1-n} Q^{\alpha\beta} \nabla_\alpha k_{\beta\mu} = X_\mu,$$

which contradicts the assumption (3.7). This proves the uniqueness of the  $SE$ -connection.

*Remark (3.3).* In virtue of the previous Theorems, we note that our  $n$ -dimensional  $SE$ -manifold  $SEX_n$  is connected by the *unique*  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  given by (3.1) and (3.6).

**Theorem (3.4).** *The following relations hold in  $SEX_n$ :*

$$(3.9) \quad K_{\omega\mu\nu} = 2\nabla_\nu k_{\omega\mu}, \quad 2\nabla_{[\omega} k_{\mu]\nu} + \nabla_\nu k_{\omega\mu} = 0.$$

**PROOF.** These assertions may be obtained from (2.9)f and (3.5).

In our subsequent considerations, we need the following two vectors:

$$(3.10) \quad S_\mu \stackrel{\text{df}}{=} S_{\mu\alpha}, \quad U_\mu \stackrel{\text{df}}{=} U_{\alpha\mu}.$$

**Theorem (3.5)** *In  $SEX_n$  the vectors  $X_\mu$ ,  $S_\mu$ , and  $U_\mu$  may be given by*

$$(3.11)a \quad X_\mu = \frac{1}{2(n-1)} K_{\mu\alpha\beta} Q^{\alpha\beta},$$

$$(3.11)b \quad S_\mu = (1-n)X_\mu = -1/2 K_{\mu\alpha\beta} Q^{\alpha\beta},$$

$$(3.11)c \quad U_\mu = k_\mu^\alpha X_\alpha = \frac{1}{2(n-1)} K_{\mu\alpha\beta} Q^{\alpha\beta}.$$

**PROOF.** Substituting for  $\nabla_\alpha k_{\beta\mu}$  into (3.6) from (3.9), we have (3.11)a. The first representations of (3.11)b, c may be obtained by contracting for  $\lambda$  and  $\mu$  in (2.8) and (2.12), respectively. The remaining representations are clear in virtue of (3.11)a and (2.9)e.

*Remark (3.6).* In a manifold connected with an Einstein connection, HLAVATÝ (1957) proved that the vector  $U_\mu$  is a gradient of a scalar  $\ln \sqrt{g}$ . That is,

$$(3.12) \quad U_\mu = 1/2 \partial_\mu (\ln g).$$

**Theorem (3.7).** *The unique  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  of  $SEX_n$  may be given by*

$$(3.13)a \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{1-n} (g_{\lambda\gamma} \nabla_\alpha k_{\beta\mu} - g_{\gamma\mu} \nabla_\alpha k_{\beta\lambda}) h^{\gamma\nu} Q^{\alpha\beta},$$

or equivalently

$$(3.13)b \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{n-1} (\delta_{[\lambda}^\nu K_{\mu]\alpha\beta} + k_{(\lambda}^\nu K_{\mu)\alpha\beta}) Q^{\alpha\beta}.$$

PROOF. In virtue of (2.2), the representation (3.1) may be written as

$$(3.14) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + (g_{\lambda\alpha} X_{\mu} - g_{\alpha\mu} X_{\lambda}) h^{\alpha\nu}.$$

Substituting for  $X_{\mu}$  into (3.14) from (3.6), we have (3.13)a. The representation (3.13)b is a consequence of (3.1) and (3.11)a.

**Theorem (3.8).** *The following five statements are equivalent in  $SEX_n$ :*

$$(a) \quad S_{\mu} = 0, \quad (b) \quad X_{\mu} = 0, \quad (c) \quad \nabla_{\omega} k_{\lambda\mu} = 0,$$

$$(d) \quad K_{\omega\lambda\mu} = 0, \quad (e) \quad \Gamma_{\lambda\mu}^{\nu} = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}.$$

PROOF. Since the tensor  $Q^{\lambda\nu}$  is of rank  $n$ , the equivalence follows from the following diagram:

$$(a) \xleftrightarrow{(3.11)b} (b) \xleftrightarrow{(3.6)} (c) \xleftrightarrow{(3.9)} (d) \xleftrightarrow{(3.13)b} (e)$$

*Remark (3.9).* Note that Theorem (3.8) is very important for the study of field equations in  $SEX_n$  since the statement (a) is one of the field equations.

#### IV. The tensor $Q^{\lambda\nu}$

In this section, we first derive a representation of the tensor  $Q^{\lambda\nu}$  in a general  $X_n$  connected by Einstein's connection and exhibit several useful representations of the unique  $n$ -dimensional  $SE$ -connection.

We need the following abbreviations for an arbitrary tensor  $X^{\lambda\nu}$ :

$$(4.1)a \quad {}^{(p)}X^{\lambda\nu} = {}^{(p)}k_{\alpha}^{\nu} X^{\lambda\alpha}, \quad (p = 0, 1, 2, \dots)$$

$$(4.1)b \quad \tilde{K}_s = 1 + K_2 + K_4 + \dots + K_s.$$

Direct calculations show that

$$(4.2)a \quad {}^{(0)}X^{\lambda\nu} = X^{\lambda\nu}, \quad {}^{(q)}k_{\alpha}^{\nu} {}^{(p)}X^{\lambda\alpha} = {}^{(p+q)}X^{\lambda\nu},$$

$$(4.2)b \quad \tilde{K}_0 = 1, \quad \tilde{K}_s + K_{s+2} = \tilde{K}_{s+2}.$$

**Theorem (4.1).** (*Recurrence relation*) *An arbitrary tensor  ${}^{(p)}X^{\lambda\nu}$  satisfies the following relation in  $SEX_n$ :*

$$(4.3)a \quad \sum_{s=0}^{n-\sigma} K_s {}^{(n-s)}X^{\lambda\nu} = 0,$$

or

$$(4.3)b \quad {}^{(n)}X^{\lambda\nu} + K_2 {}^{(n-2)}X^{\lambda\nu} + K_4 {}^{(n-4)}X^{\lambda\nu} + \dots \\ \dots + K_{n-\sigma-2} {}^{(\sigma+2)}X^{\lambda\nu} + K_{n-\sigma} {}^{(\sigma)}X^{\lambda\nu} = 0.$$

PROOF. Multiplying  $X^{\lambda\alpha}$  to both sides of (2.10)e, we get have (4.3).

Now, we are ready to derive a representation of  $Q^{\lambda\nu}$ .

**Theorem (4.2).** In  $X_n$  the tensor  $Q^{\lambda\nu}$  may be given by

$$(4.4)a \quad Q^{\lambda\nu} = -\frac{1}{g} M^{\lambda\nu},$$

or

$$(4.4)b \quad Q^{\lambda\nu} = \begin{cases} -\frac{1}{g} \left( {}^{(n-2)}k^{\lambda\nu} + \tilde{K}_2 {}^{(n-4)}k^{\lambda\nu} + \tilde{K}_4 {}^{(n-6)}k^{\lambda\nu} + \dots \right. \\ \quad \left. \dots + \tilde{K}_{n-4} {}^{(2)}k^{\lambda\nu} + \tilde{K}_{n-2} h^{\lambda\nu} \right), & \text{if } n \text{ is even} \\ -\frac{1}{g} \left( {}^{(n-1)}k^{\lambda\nu} + \tilde{K}_2 {}^{(n-3)}k^{\lambda\nu} + \tilde{K}_4 {}^{(n-5)}k^{\lambda\nu} + \dots \right. \\ \quad \left. \dots + \tilde{K}_{n-3} {}^{(2)}k^{\lambda\nu} + \tilde{K}_{n-1} h^{\lambda\nu} \right), & \text{if } n \text{ is odd,} \end{cases}$$

where  $M^{\lambda\nu}$  is a symmetric tensor defined by

$$(4.5) \quad M^{\lambda\nu} \stackrel{\text{df}}{=} \sum_{s=0}^{n-1} \tilde{K}_s {}^{(n-2+\sigma-s)}k^{\lambda\nu}.$$

PROOF. Consider a tensor  $X^{\lambda\nu}$  defined by

$$(4.6) \quad P_{\alpha\mu} X^{\alpha\nu} \stackrel{\text{df}}{=} \delta_\mu^\nu.$$

Since the symmetric tensor  $P_{\lambda\mu}$  is of rank  $n$ , the tensor  $X^{\lambda\nu} = X^{\nu\lambda}$  exists uniquely and equals  $Q^{\lambda\nu}$  in virtue of (3.4). Hence it suffices to find a representation of the tensor  $X^{\lambda\nu}$ . We first note that the tensor  $X^{\lambda\nu}$  satisfies the following recurrence relations:

$$(4.7) \quad {}^{(p)}X^{\lambda\nu} = {}^{(p-2)}X^{\lambda\nu} + {}^{(p-2)}k^{\lambda\nu}, \quad (p \geq 2),$$

which may be obtained multiplying both sides of (4.6) by  ${}^{(p-2)}k^{\mu\lambda}$  and making use of (3.3) and (4.1)a.

In order to derive (4.4), we now substitute for  ${}^{(n)}X^{\lambda\nu}$  into (4.3)b from (4.7) and make use of (4.1)b to obtain

$$(4.8)a \quad \tilde{K}_0 {}^{(n-2)}k^{\lambda\nu} + \tilde{K}_2 {}^{(n-4)}k^{\lambda\nu} + \tilde{K}_4 {}^{(n-6)}k^{\lambda\nu} + \dots + K_{n-\sigma} {}^{(\sigma)}X^{\lambda\nu} = 0.$$

Substituting again for  ${}^{(n-2)}X^{\lambda\nu}$  into (4.8) a from (4.7), we have

$$(4.8)b \quad \tilde{K}_0 {}^{(n-2)}k^{\lambda\nu} + \tilde{K}_2 {}^{(n-4)}k^{\lambda\nu} + \tilde{K}_4 {}^{(n-6)}k^{\lambda\nu} + \dots + K_{n-\sigma} {}^{(\sigma)}X^{\lambda\nu} = 0.$$

After  $\frac{n-\sigma}{2}$  steps of repeated substitutions for  ${}^{(p)}X^{\lambda\nu}$ , we have

$$(4.8)c \quad \tilde{K}_0 {}^{(n-2)}k^{\lambda\nu} + \tilde{K}_2 {}^{(n-4)}k^{\lambda\nu} + \tilde{K}_4 {}^{(n-6)}k^{\lambda\nu} + \dots + \tilde{K}_{n-\sigma-4} {}^{(2+\sigma)}k^{\lambda\nu} + \tilde{K}_{n-\sigma-2} {}^{(\sigma)}k^{\lambda\nu} + K_{n-\sigma} {}^{(\sigma)}X^{\lambda\nu} = 0.$$

Now, multiply both sides of (4.8)c by  ${}^{(\sigma)}k_\nu^\mu$  to obtain

$$(4.8)d \quad \tilde{K}_0 {}^{(n-2+\sigma)}k^{\lambda\nu} + \tilde{K}_2 {}^{(n-4+\sigma)}k^{\lambda\nu} + \tilde{K}_4 {}^{(n-6+\sigma)}k^{\lambda\nu} + \dots + \tilde{K}_{n-\sigma-4} {}^{(2+2\sigma)}k^{\lambda\nu} + \tilde{K}_{n-\sigma-2} {}^{(2\sigma)}k^{\lambda\nu} + K_{n-\sigma} {}^{(2\sigma)}X^{\lambda\nu} = 0.$$

Substituting

$${}^{(2\sigma)}X^{\lambda\nu} = {}^{(0)}X^{\lambda\nu} + \sigma {}^{(0)}k^{\lambda\nu}$$



into (4.8)d and making use of (2.9)c, (2.10)d, and (4.2), we finally get have a representation (4.4)b for the tensor  $X^{\lambda\nu} = Q^{\lambda\nu}$ . (4.4)b can be easily summarized as (4.4)a. Note, in particular, that the tensor  ${}^{(n-2+\sigma-s)}k^{\lambda\nu}$  is symmetric according to (2.10)a since  $n-2+\sigma-s$  is an even integer. Hence the tensor  $M^{\lambda\nu}$  is also symmetric.

*Remark (4.3).* We note that according to Chung's previous results (Chung, 1981, 1985),

$$(4.9) \quad P_{\lambda\mu} = -{}^*h_{\lambda\mu} = -{}^*g_{(\lambda\mu)}, \quad Q^{\lambda\nu} = -{}^*h^{\lambda\nu} = -{}^*g^{(\lambda\nu)}.$$

Therefore, the tensors  $P_{\lambda\mu}$  and  $Q^{\lambda\nu}$  are both of rank  $n$ .

The next two Theorems are consequences of Theorem (4.2). They may be obtained from (3.11) and (3.13) making use of (3.9).

**Theorem (4.4).** In  $SEX_n$  the vectors  $X_\mu$ ,  $S_\mu$ , and  $U_\mu$  may be given by

$$(4.10)a \quad X_\mu = \frac{1}{(n-1)g} M^{\alpha\beta} \nabla_\alpha k_{\beta\mu} = \frac{1}{2(1-n)g} K_{\mu\alpha\beta} M^{\alpha\beta},$$

$$(4.10)b \quad S_\mu = -\frac{1}{g} M^{\alpha\beta} \nabla_\alpha k_{\beta\mu} = \frac{1}{2g} K_{\mu\alpha\beta} M^{\alpha\beta},$$

$$(4.10)c \quad U_\mu = \frac{1}{(n-1)g} M^{\alpha\beta} k_\mu^\gamma \nabla_\alpha k_{\beta\gamma} = \frac{1}{2(1-n)g} K_{\mu\alpha\beta} M^{\alpha\beta}.$$

**Theorem (4.5).** The unique  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  may be given by the following expressions:

$$(4.11)a \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{(n-1)g} (g_{\lambda\gamma} \nabla_\alpha k_{\beta\mu} - g_{\gamma\mu} \nabla_\alpha k_{\beta\lambda}) h^{\nu\gamma} M^{\alpha\beta},$$

$$(4.11)b \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{2(1-n)g} (g_{\lambda\gamma} K_{\mu\alpha\beta} - g_{\gamma\mu} K_{\lambda\alpha\beta}) h^{\nu\gamma} M^{\alpha\beta},$$

$$(4.11)c \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{2}{(n-1)g} ((\nabla_\alpha k_{\beta[\mu}) \delta_{\lambda]}^\nu + (\nabla_\alpha k_{\beta(\lambda}) k_{\mu)}^\nu) M^{\alpha\beta},$$

$$(4.11)d \quad \Gamma_{\lambda\mu}^\nu = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + \frac{1}{(1-n)g} (\delta_{[\lambda}^\nu K_{\mu]\alpha\beta} + k_{(\lambda}^\nu K_{\mu)\alpha\beta}) M^{\alpha\beta}.$$

*Remark (4.6).* In the following table we illustrate the representations of  $g$  and the tensor  $M^{\lambda\nu}$  for the lower dimensional cases, which may be used to calculate the corresponding representations of the vectors  $X_\mu$ ,  $S_\mu$ , and  $U_\mu$  and the  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  for  $n=2, 3, 4$ .

| $n$ | $g$ |           | $M^{\lambda\nu}$                              |  |
|-----|-----|-----------|-----------------------------------------------|--|
|     |     |           |                                               |  |
| 2   | 1   | $k$       | $h^{\lambda\nu}$                              |  |
| 3   | 1   | $K_2$     | $gh^{\lambda\nu} + {}^{(2)}k^{\lambda\nu}$    |  |
| 4   | 1   | $K_2 + k$ | $K_2 h^{\lambda\nu} + {}^{(2)}k^{\lambda\nu}$ |  |

(4.12)



### V. Conformal change of $SEX_n$

In this final section we investigate the change  $\Gamma_{\lambda\mu}^\nu \rightarrow \bar{\Gamma}_{\lambda\mu}^\nu$  of the  $SE$ -connection induced by a conformal change of the tensor  $g_{\lambda\mu}$ .

Consider two  $n$ -dimensional  $SE$ -manifolds  $SEX_n(\overline{SEX}_n)$ , on which the differential geometric structure is imposed by a general real tensor  $g_{\lambda\mu}$  ( $\bar{g}_{\lambda\mu}$ ) through the  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  ( $\bar{\Gamma}_{\lambda\mu}^\nu$ ) given by (4.11) ((5.1)):

$$(5.1) \quad \bar{\Gamma}_{\lambda\mu}^\nu = \left\{ \bar{\nu} \right\}_{\lambda\mu} + \frac{1}{(1-n)\bar{g}} (\delta_{[\lambda}^{\nu} \bar{K}_{\mu]\alpha\beta} + \bar{K}_{(\lambda}^{\nu} \bar{K}_{\mu)\alpha\beta}) \bar{M}^{\alpha\beta}.$$

We say that  $SEX_n$  and  $\overline{SEX}_n$  are *conformal* if, and only if

$$(5.2) \quad \bar{g}_{\lambda\mu}(x) = e^{\Omega} g_{\lambda\mu}(x),$$

where  $\Omega = \Omega(x)$  is an at least twice differentiable function. This conformal change enforces a change of the connection. An explicit representation for  $\bar{\Gamma}_{\lambda\mu}^\nu$  will be exhibited in this section.

*Agreement (5.1).* Throughout this section, we agree that, if  $T$  is a function of  $g_{\lambda\mu}$ , then we denote by  $\bar{T}$  the same function of  $\bar{g}_{\lambda\mu}$ . In particular, if  $T$  is a tensor, so is  $\bar{T}$ . Furthermore, the indices of  $T$  ( $\bar{T}$ ) will be raised and/or lowered by means of  $h^{\lambda\nu}$  ( $\bar{h}^{\lambda\nu}$ ) and/or  $h_{\lambda\mu}$  ( $\bar{h}_{\lambda\mu}$ ).

The results listed in the following Theorem are immediate consequences of (5.2) and Agreement (5.1).

**Theorem (5.2).** *The conformal change (5.2) induces the following changes:*

$$(5.3)a \quad {}^{(p)}\bar{K}_{\lambda\mu} = e^{\Omega(p)} K_{\lambda\mu}, \quad {}^{(p)}\bar{K}^{\lambda\nu} = e^{-\Omega(p)} K^{\lambda\nu}, \quad {}^{(p)}\bar{K}_\lambda^\nu = {}^{(p)}K_\lambda^\nu;$$

$$(5.3)b \quad \bar{h} = e^{\Omega} h, \quad \bar{t} = e^{\Omega} t, \quad \bar{g} = e^{\Omega} g, \quad \bar{g} = g, \quad \bar{K}_p = K_p, \\ (p = 0, 1, 2, \dots)$$

**Theorem (5.3).** *The conformal change (5.2) induces the following changes:*

$$(5.4)a \quad \left\{ \bar{\nu} \right\}_{\lambda\mu} = \left\{ \nu \right\}_{\lambda\mu} + \delta_{(\lambda}^{\nu} \Omega_{\mu)} - 1/2 h_{\lambda\mu} h^{\alpha\beta} \Omega_{\alpha\beta}, \quad (\Omega_\mu \stackrel{\text{df}}{=} \partial_\mu \Omega)$$

$$(5.4)b \quad \bar{\nabla}_\nu \bar{K}_{\omega\mu} = e^{\Omega} (\nabla_\nu K_{\omega\mu} + K_{\nu[\omega} \Omega_{\mu]} - h_{\nu[\omega} K_{\mu]}^\alpha \Omega_\alpha),$$

$$(5.4)c \quad \bar{K}_{\omega\mu\nu} = e^{\Omega} (K_{\omega\mu\nu} - K_{\omega\mu} \Omega_\nu - 2h_{\nu[\omega} K_{\mu]}^\alpha \Omega_\alpha),$$

$$(5.4)d \quad \bar{M}^{\lambda\nu} = e^{-\Omega} M^{\lambda\nu}.$$

**PROOF.** The first relation follows by substituting (5.3)a into

$$\left\{ \bar{\nu} \right\}_{\lambda\mu} = \bar{h}^{\nu\alpha} (\partial_{(\mu} \bar{h}_{\lambda)\alpha} - 1/2 \partial_\alpha \bar{h}_{\lambda\mu}).$$

Substituting (5.3)a and (5.4)a into

$$\bar{\nabla}_\nu \bar{K}_{\omega\mu} = \partial_\nu \bar{K}_{\omega\mu} - \bar{K}_{\alpha\mu} \left\{ \bar{\alpha} \right\}_{\omega\nu} - \bar{K}_{\omega\alpha} \left\{ \bar{\alpha} \right\}_{\mu\nu},$$

we have (5.4)b. (5.4)c may be obtained by substituting (5.4)b into the corresponding representation for  $\bar{K}_{\omega\mu\nu}$  in (2.9)f. The last relation is obvious in virtue of (4.5) and (5.3).

**Theorem (5.4).** *If  $SEX_n$  and  $\overline{SEX}_n$  are conformal, the function  $\Omega$  satisfies*

$$(5.5) \quad 2k_{v[\omega}\Omega_{\mu]} + k_{\omega\mu}\Omega_v = 0.$$

PROOF. Since  $SEX$  and  $\overline{SEX}_n$  are  $SE$ -manifolds, we have

$$(5.6) \quad \bar{K}_{\omega\mu\nu} = 2\bar{\nabla}_\nu \bar{k}_{\omega\mu}, \quad K_{\omega\mu\nu} = 2\nabla_\nu k_{\omega\mu},$$

in virtue of Theorem (3.4). Our assertion follows from (5.6), (5.4)b, and (5.4)c.

*Remark (5.5).* In virtue of (5.5), we may simplify the relation (5.4)b as

$$(5.7) \quad \bar{\nabla}_\nu \bar{k}_{\omega\mu} = e^\Omega (\nabla_\nu k_{\omega\mu} - 1/2 \Omega_\nu k_{\omega\mu} - h_{v[\omega} k_{\mu]}^\alpha \Omega_\alpha).$$

Now, we are ready to derive representations of the changes  $X_\mu \rightarrow \bar{X}_\mu$  and  $\Gamma_{\lambda\mu}^\nu \rightarrow \bar{\Gamma}_{\lambda\mu}^\nu$  induced by the conformal change (5.2).

**Theorem (5.6).** *The vector  $X_\mu$  is transformed by the conformal change (5.2) as follows:*

$$(5.8)a \quad \bar{X}_\mu = X_\mu + \frac{1}{(n-1)g} M^{\alpha\beta} (1/2 k_{\mu\beta} \Omega_\alpha - h_{\alpha[\beta} k_{\mu]}^\gamma \Omega_\gamma),$$

or equivalently

$$(5.8)b \quad \bar{X}_\mu = X_\mu + \frac{1}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s^{(s)} Z_\mu,$$

where

$$(5.9) \quad \tilde{Z}_\mu^{(s)} \stackrel{\text{df}}{=} \binom{n-1+\sigma-s}{s} k_\mu^\alpha - 1/2 \binom{n-2+\sigma-s}{s} k_\beta^\alpha k_\mu^\beta \Omega_\alpha.$$

PROOF. In virtue of (4.10)a and Agreement (5.1), we have

$$(5.10) \quad \bar{X}_\mu = \frac{1}{(n-1)\bar{g}} \bar{M}^{\alpha\beta} \bar{\nabla}_\alpha \bar{k}_{\beta\mu}.$$

The relation (5.8)a may be obtained by substituting (5.3)b, (5.4)d, and (5.7) into (5.10). The second relation follows from (5.8)a, using (2.9)c, (4.5), and (5.9). Note that the tensor  $\binom{n-2+\sigma-s}{s} k^{\alpha\beta}$  is symmetric.

**Theorem (5.7).** *The  $SE$ -connection  $\Gamma_{\lambda\mu}^\nu$  is transformed by the conformal change (5.2) as follows:*

$$(5.11) \quad \begin{aligned} \bar{\Gamma}_{\lambda\mu}^\nu &= \Gamma_{\lambda\mu}^\nu + \delta_{(\lambda}^\nu \Omega_{\mu)} - 1/2 h_{\lambda\mu} h^{\nu\alpha} \Omega_\alpha + \\ &+ \frac{2}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s^{(s)} (\delta_{[\lambda}^\nu \tilde{Z}_{\mu]}^{(s)} + k_{(\lambda}^\nu \tilde{Z}_{\mu]}^{(s)}). \end{aligned}$$

PROOF. In virtue of (3.1), (5.4)a, and (5.8)b, the relation (5.11) may be derived as follows:

$$\begin{aligned} \bar{\Gamma}_{\lambda\mu}^{\nu} &= \left\{ \bar{\nu} \right\}_{\lambda\mu} + 2\delta_{[\lambda}^{\nu} \bar{X}_{\mu]} + 2\bar{k}_{[\lambda}^{\nu} \bar{X}_{\mu]} = \left\{ \nu \right\}_{\lambda\mu} + \delta_{[\lambda}^{\nu} \Omega_{\mu]} - 1/2 h_{\lambda\mu} h^{\nu\alpha} \Omega_{\alpha} + \\ &+ 2\delta_{[\lambda}^{\nu} X_{\mu]} + \frac{2}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s \delta_{[\lambda}^{\nu} Z_{\mu]}^{(s)} + 2k_{[\lambda}^{\nu} X_{\mu]} + \frac{2}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s k_{[\lambda}^{\nu} Z_{\mu]}^{(s)} = \\ &= (\text{The right-hand side of (5.11)}). \end{aligned}$$

*Remark (5.8).* The relation (5.11) may be derived directly from (4.11), using (4.5) and (5.4).

*Remark (5.9).* As direct consequences of the above two theorems, we still state the representations of the changes of tensors  $S_{\lambda\mu}^{\nu}$ ,  $U_{\lambda\mu}^{\nu}$ , and vectors  $S_{\mu}$  and  $U_{\mu}$  induced by the conformal change (5.2):

$$(5.12)a \quad \bar{S}_{\lambda\mu}^{\nu} = S_{\lambda\mu}^{\nu} + \frac{2}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s \delta_{[\lambda}^{\nu} Z_{\mu]}^{(s)},$$

$$(5.12)b \quad \bar{U}_{\lambda\mu}^{\nu} = U_{\lambda\mu}^{\nu} + \frac{2}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s k_{[\lambda}^{\nu} Z_{\mu]}^{(s)},$$

$$(5.12)c \quad \bar{S}_{\mu} = S_{\mu} - \frac{1}{g} \sum_{s=0}^{n-1} \tilde{K}_s Z_{\mu}^{(s)},$$

$$(5.12)d \quad \bar{U}_{\mu} = U_{\mu} + \frac{1}{(n-1)g} \sum_{s=0}^{n-1} \tilde{K}_s k_{\mu}^{\alpha} Z_{\alpha}^{(s)}.$$

*Remark (5.10).* The following table, which illustrates the tensor  $Z_{\mu}^{(s)}$  for  $n=2, 3, 4$ , may be used to calculate various representations of changes given in (5.8)b, (5.11), and (5.12) for the lower dimensional cases.

| $n \backslash \begin{matrix} (s) \\ Z_{\mu} \end{matrix}$ | $\begin{matrix} (0) \\ Z_{\mu} \end{matrix}$                      | $\begin{matrix} (2) \\ Z_{\mu} \end{matrix}$ |
|-----------------------------------------------------------|-------------------------------------------------------------------|----------------------------------------------|
| 2                                                         | 0                                                                 | 0                                            |
| 3                                                         | $(^{(3)}k_{\mu}^{\alpha} + K_2 k_{\mu}^{\alpha}) \Omega_{\alpha}$ | $-1/2 k_{\mu}^{\alpha} \Omega_{\alpha}$      |
| 4                                                         | $(^{(3)}k_{\mu}^{\alpha} + K_2 k_{\mu}^{\alpha}) \Omega_{\alpha}$ | $-2 k_{\mu}^{\alpha} \Omega_{\alpha}$        |

According to this table and (5.11), we have for  $n=2$

$$(5.14) \quad \bar{\Gamma}_{\lambda\mu}^{\nu} = \Gamma_{\lambda\mu}^{\nu} + \delta_{[\lambda}^{\nu} \Omega_{\mu]} - 1/2 h_{\lambda\mu} h^{\nu\alpha} \Omega_{\alpha}.$$

As an immediate consequence of (5.14), we have

**Theorem (5.11).** *In a 2-dimensional  $SE$ -manifold  $SEX_2$ , the tensors  $S_{\lambda\mu}^v$  and  $U_{\lambda\mu}^v$  are conformal invariant with respect to (5.2). That is,*

$$(5.15) \quad \bar{S}_{\lambda\mu}^v = S_{\lambda\mu}^v, \quad \bar{U}_{\lambda\mu}^v = U_{\lambda\mu}^v.$$

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