On generalized continued fraction expansions of short period length

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Abstract, We present infinite classes of totally complex quartic orders in which the generalized Voronoi expansions have period length 1, 2, 4 and formulae for the fundamental units in those orders.

1. Introduction

In [1] the author presented a generalization of the Voronoi continued fraction algorithm. The expansions of the generalized Voronoi algorithm (GVA) are periodic in every order of an arbitrary algebraic number field.

Since it is well known that there are infinitely many quadratic irrationalities whose continued fraction expansions have a prescribed period length, the question arises whether an analogous result is true for the GVA. In this paper we present infinitely many totally complex quartic orders in which the GVA has period length 1, 2 and 4. Moreover, we give formulae for the fundamental units of these orders.

Similar results for Voronoi's algorithm in complex cubic fields are due to DUBOIS [2] and WILLIAMS [5].

2. Preliminaries

In this paper we study the order

$$\mathcal{O} = \mathbf{Z}[\sqrt[4]{-D}]$$

in the algebraic number field

$$F = \mathbf{Q}(\sqrt[4]{-D}).$$

The positive integer D is of the form

$$D = 4k^4+d,$$

* Supported by a Feodor Lynen research fellowship of the Alexander von Humboldt Foundation.

where k is a positive integer and d is an integer satisfying

$$0 < |d| \leqq 4k.$$

Then F is a totally complex quartic field.

We fix

$$\theta = \sqrt[4]{-D} = (1+i)\varkappa$$

where

$$\varkappa = (D/4)^{1/4} \in \mathbf{R}.$$

The map

$$\sigma\colon\ \theta \longrightarrow -\theta$$

is the automorphism of F which keeps the subfield

$$K = \mathbf{Q}(\theta^2)$$

fixed. For $\xi \in K$ we write $\xi^{(1)} = \xi$, $\xi^{(2)} = \sigma(\xi)$. For a real number r we denote by $[r]$ the greatest integer smaller than r and by $\langle r \rangle$ the nearest integer to r.

3. The generalized Voronoi algorithm

First of all, we briefly recall the GVA in $\mathcal{O}$, cf. [1]. By

$$F \to \mathbf{R}^2$$

$$\zeta \mapsto \underline{\zeta} = (|\zeta|^2,\ |\sigma(\zeta)|^2)^T$$

we map the field F into the plane $\mathbf{R}^2$. For every point $\vec{y} = (y_1, y_2)^T$ in the first quadrant of the plane we call

$$Q(\vec{y}) = \{\vec{y}' \in \mathbf{R}^2 | 0 \leqq y_k' \leqq y_k,\ 1 \leqq k \leqq 2\}$$

the *norm body* of $\vec{y}$.

Now, $\underline{\mathcal{O}}$ is a discrete set in $\mathbf{R}^2$. In $\underline{\mathcal{O}} \setminus \{\underline{0}\}$ we call those points $\underline{\mu}$ *minimal points* whose norm body does not contain points of $\underline{\mathcal{O}}$ aside from $\underline{0}$ and $\underline{\mu}$.

If $\underline{\mu}$ is a minimal point and $\{u, v\} = \{1, 2\}$ then the u-neighbor of $\underline{\mu}$ is the uniquely determined minimal point $\underline{\mu}'$ with

$$|\mu'^{(v)}| < |\mu^{(v)}| \tag{3.1}$$

and $|\mu'^{(u)}|$ minimal.

Starting with the minimal point $\underline{\mu}_0 = \underline{1}$ we can form a two-sided chain of minimal points $\{\underline{\mu}_k\}_{k \in \mathbf{Z}}$ with the property that $\underline{\mu}_{k+1}$ is always the 2-neighbor of $\underline{\mu}_k$, whereas $\underline{\mu}_k$ is the 1-neighbor of $\underline{\mu}_{k+1}$.

Moreover, we have for every $k \in \mathbf{Z}$

$$\underline{\mu}_{-k} = \sigma(\underline{\mu}_k). \tag{3.2}$$

The chain is of the purely periodic form

$$\underline{\varepsilon^{-1}}, \underline{\varepsilon^{-1}\mu_1}, \ldots, \underline{\varepsilon^{-1}\mu_{p-1}}, \underline{1}, \ldots, \underline{\mu_{p-1}}, \underline{\varepsilon}, \underline{\varepsilon\mu_1}, \ldots \tag{3.3}$$

where ε is a fundamental unit of $\mathcal{O}$, when chosen minimal the number p is called the *period length* of the GVA in $\mathcal{O}$.

4. The main theorems

We fix

$$\varepsilon = 2k^2+2k\theta+\theta^2. \tag{4.1}$$

Theorem (4.1). *If $d=1$, then the period length of the* GVA *in $\mathcal{O}$ is 1 and ε is a fundamental unit of $\mathcal{O}$.*

Theorem (4.2). *If $d>1$ and $d|4k$, then the period length of the* GVA *in $\mathcal{O}$ is 2 and ε^2/d is a fundamental unit of $\mathcal{O}$.*

Theorem (4.3). *If $k\geqq 2$ and $d=-1$, then the period length of the* GVA *in $\mathcal{O}$ is 2 and ε is a fundamental unit of $\mathcal{O}$.*

Theorem (4.4). *If $k\geqq 2$, $d<-1$ and $d|4k$, then the period length of the* GVA *in $\mathcal{O}$ is 4 and ε^2/d is a fundamental unit of $\mathcal{O}$.*

5. The proofs

In order to be able to demonstrate the theorems of the previous section, we prove some lemmata. We always assume in this section, that α is a number in $\mathcal{O}$, given in the representation

$$\alpha = a_0+a_1\theta+a_2\theta^2+a_3\theta^3, \tag{5.1}$$

with integral coefficients $a_0, \ldots, a_3$.

Lemma (5.2). *If $|\sigma(\alpha)|<1$ and $a_3\neq 0$, then $|\alpha|^2\geqq(4\varkappa^3-1)^2$.*

PROOF. We can write α and $\sigma(\alpha)$ in the form

$$\alpha = (a_0+a_1\varkappa-2a_3\varkappa^3)+i(a_1\varkappa+2a_2\varkappa^2+2a_3\varkappa^3), \tag{5.3}$$

$$\sigma(\alpha) = (a_0-a_1\varkappa+2a_3\varkappa^3)+i(-a_1\varkappa+2a_2\varkappa^2-2a_3\varkappa^3). \tag{5.4}$$

Now we set

$$A = a_1\varkappa-2a_3\varkappa^3 \quad \text{and} \quad B = a_1\varkappa+2a_3\varkappa^3$$

and obtain

$$\begin{aligned} |\alpha|^2 &= (a_0+A)^2+(2a_2\varkappa^2+B)^2, \\ |\sigma(\alpha)|^2 &= (a_0-A)^2+(2a_2\varkappa^2-B)^2 < 1. \end{aligned} \tag{5.5}$$

On the other hand, since $a_3\neq 0$, we have $|A|\geqq 2\varkappa^3$ or $|B|\geqq 2\varkappa^3$ and therefore $\operatorname{sgn} A=\operatorname{sgn} a_0$ or $\operatorname{sgn} B=\operatorname{sgn} a_2$. Hence, our statement follows from (5.5). □

Lemma (5.6). *We have* $|\varkappa - k| < 1/(2k^2)$.

PROOF. We know that

$$|d| = 4|\varkappa^4 - k^4| = 4|\varkappa - k||\varkappa^3 + \varkappa^2 k + \varkappa k^2 + k^3|.$$

Hence, it follows from our assumption that

$$|\varkappa - k| < \frac{k}{4(\min\{\varkappa, k\})^3}. \tag{5.7}$$

If $k < \varkappa$, then our statement follows immediately from (5.7).

If $k > \varkappa$, then we have

$$|\varkappa - k| < \frac{\varkappa k}{4(k^4 - k)} < \frac{k^2}{4(k^4 - k)},$$

and this proves our statement for $k \geqq 2$. For $k = 1$, the inequality follows from an easy computation.

Corollary (5.8). *We have*

$$\langle 2\varkappa \rangle = 2k \quad \text{and} \quad \langle 2k\varkappa \rangle = 2k^2.$$

Lemma (5.9). *If* $0 < d \leqq 4k$, *then* $\underline{\varepsilon}$ *is the 2-neighbor of* $\underline{1}$ *in* $\underline{\mathcal{O}}$.

PROOF. We notice that

$$\varepsilon = 2(k + \varkappa)(k + i\varkappa) \quad \text{and} \quad \sigma(\varepsilon) = 2(k - \varkappa)(k - i\varkappa). \tag{5.10}$$

Hence,

$$|\varepsilon|^2 < 32\varkappa^4, \tag{5.11}$$

and by Lemma (5.6)

$$|\sigma(\varepsilon)|^2 < 1. \tag{5.12}$$

Now let $\underline{\alpha}$ be the 2-neighbor of $\underline{1}$ in $\mathcal{O}$. Then it follows from (5.11) and (5.12) that $|\alpha|^2 < 32\varkappa^4$. Hence, by Lemma (5.2) we must have $a_3 = 0$ for $D \geqq 64$ and the same is true for $D < 64$, as our computations show. Without loss of generality, we assume that $a_2 > 0$. Since $|\sigma(\alpha)| < 1$, we must have

$$a_1 \geqq [2a_2\varkappa] \quad \text{and} \quad a_0 \geqq [a_1\varkappa].$$

Therefore, the choice $a_2 = 1$, $a_1 = 2k$ and $a_0 = 2k^2$ makes $|\alpha|^2$ minimal.

Lemma (5.13). *If* $k \geqq 2$ *and* $-4k \leqq d < 0$, *then* $\underline{\varepsilon - 1}$ *is the 2-neighbor of* $\underline{1}$ *in* $\underline{\mathcal{O}}$.

PROOF. Since $k - \varkappa > 0$, it follows from Lemma (5.6) that

$$|\sigma(\varepsilon - 1)|^2 < 1. \tag{5.14}$$

Moreover, we have

$$|\varepsilon - 1|^2 < |\varepsilon|^2. \tag{5.15}$$

If $\underline{\alpha}$ is the 2-neighbor of $\underline{1}$ in $\mathcal{O}$, then it follows as in the proof of the previous lemma that $a_3=0$. For $k\geqq 2$ we have by Lemma (5.6) $\varkappa\geqq 1$, and therefore we can assume that $a_2>0$. Now it follows from $|\sigma(\alpha)|<1$ that

$$[2a_2\varkappa] \leqq a_1 \leqq [2a_2\varkappa]+1 \tag{5.16}$$

and

$$[a_1\varkappa] \leqq a_0 \leqq [a_1\varkappa]+1. \tag{5.17}$$

If $a_2\geqq 2$, then $|\alpha|^2\geqq 32\varkappa^4$, which is impossible by (5.11), (5.14) and (5.15). Hence we have $a_2=1$. By Corollary (5.8) and (5.16) this means that either $a_1=2k-1$ or $a_1=2k$, but if $a_1=2k-1$, then we have by Lemma (5.6)

$$|\sigma(\alpha)|^2 > \varkappa^2(1-2(k-\varkappa))^2 > 1.$$

Thus, $a_1=2k$ and the choice $a_0=2k^2$ makes $|\alpha|^2$ minimal.

We get immediately

Corollary (5.18). *If $k\geqq 2$ and $-4k\leqq d<0$, then $\underline{\varepsilon}$ is the 2-neighbor of $\underline{\varepsilon-1}$ in $\underline{\mathcal{O}}$.*

Finally, an easy computation shows

Lemma (5.19).

$$N_{F|K}(\varepsilon) = -d.$$

Now we can prove our theorems. First of all, we discuss the case $d>0$.

From (3.2) and Lemma (5.9) we learn that

$$\ldots, \underline{\sigma(\varepsilon)}, \underline{1}, \underline{\varepsilon}, \ldots \tag{5.20}$$

is a part of the GVA expansion in $\mathcal{O}$. Hence, Lemma (5.19) shows that the period length is 1 for $d=1$. *This proves Theorem* (4.1).

If $d>1$, then by Lemma (5.19) the period length has to be at least 2. Now we have

$$\varepsilon/\sigma(\varepsilon) = \varepsilon^2/(-d) = (d+8k^3\theta+8k^2\theta^2+4k\theta^3)/(-d) \tag{5.21}$$

and Theorem (4.2) follows from (5.20) and (5.21).

Now let $d<0$ and $k\geqq 2$.

Then by (3.2) ,Lemma (5.13) and Corollary (5.18)

$$\ldots, \underline{\sigma(\varepsilon)}, \underline{\sigma(\varepsilon-1)}, \underline{1}, \underline{\varepsilon-1}, \underline{\varepsilon}, \ldots \tag{5.22}$$

is a part of the GVA expansion in $\mathcal{O}$.

Since

$$N_{F|K}(\varepsilon-1) = 2(1-2k^2-\theta^2)-d$$

the period length is at least 2 and it follows from Lemma (5.19) for $d=-1$ that the period length is 2. This proves Theorem (4.3). Now it follows from (4.21) that

$$(\varepsilon-1)/\sigma(\varepsilon) = (\varepsilon^2-\varepsilon)/(-d) \notin O$$

if $d|4k$, but that $\varepsilon/\sigma(\varepsilon)$ is a unit in this case. This proves Theorem (4.4).

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(Received October 17, 1986)