

The Pexider equation on n -semigroups and n -groups

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In this paper we describe a certain family of solutions of the Pexider equation on n -semigroups $S(\)$ and $T[\]$ possessing invertible elements. If $T[\]$ is an n -group we give a general solution of the Pexider equation. Moreover, the above mentioned results are used to describe a certain family of solutions of another functional equation of the form:

$$F_1(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1) = F_{n+1}(x, (s_1, s_2, \dots, s_n)).$$

The Pexider equation on n -semigroups (n -groupoids) is a straight-forward generalization of the Pexider equation on various algebraic structures with binary operations (cf. A. KRAPEŽ and M. A. TAYLOR [5]). Algebraically, the Pexider equation is related to the notions of homotopy and of isotopy.

We begin with some definitions.

A Pexider equation on binary groupoids S and T is a functional equation

$$(1) \quad \alpha_3(s_1 s_2) = \alpha_1(s_1) \alpha_2(s_2)$$

for arbitrary $s_1, s_2 \in S$, where $\alpha_1, \alpha_2, \alpha_3: S \rightarrow T$ are unknown functions.

A triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ being a solution of equation (1) is called a homotopy from the groupoid S into the groupoid T . If $\alpha_1, \alpha_2, \alpha_3$ are bijections, then the homotopy $(\alpha_1, \alpha_2, \alpha_3)$ is called an isotopy from the groupoid S onto the groupoid T .

Let $a_1, a_2 \in T$ be arbitrary fixed elements of the groupoid T . Functions $L_{a_1}(t) = a_1 t$ and $R_{a_2}(t) = t a_2$ for $t \in T$ we call a left translation and a right translation on the groupoid T , respectively.

An element $t \in T$ is called an invertible element in the semigroup T if $tT = Tt = T$. The symbol $R(T)$ will denote the set of all invertible elements in the semigroup T . It is known that if $R(T) \neq \emptyset$, then $R(T)$ is a subgroup of the semigroup T and the identity of the subgroup $R(T)$ is an identity of the semigroup T and so T is a monoid (cf. [6]).

An element $t \in T$ is called a right-cancellative element in the semigroup T if

$$\forall t_1, t_2 \in T [t_1 t = t_2 t \Rightarrow t_1 = t_2].$$

Theorem 1. *Let S be a groupoid with identity, and let T be a semigroup. If a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ is a solution of equation (1) on S and T such that $\alpha_1(1)$ and $\alpha_2(1)$*

are elements from $R(T)$, then there exists a homomorphism $\varphi: S \rightarrow T$ and there exist elements $a_1, a_2 \in R(T)$ such that

$$(2) \quad \alpha_1 = L_{a_1}\varphi, \quad \alpha_2 = R_{a_2}\varphi, \quad \alpha_3 = L_{a_1}R_{a_2}\varphi.$$

If $\varphi: S \rightarrow T$ is an arbitrary homomorphism and $a_1, a_2 \in T$ are arbitrary elements, then a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (2) is a solution of equation (1).

PROOF. Denote $a_1 := \alpha_1(1)$ and $a_2 := \alpha_2(1)$. Let us put $\varphi(s) := a_1^{-1}\alpha_1(s)$ for every $s \in S$. Notice that $\alpha_3(s) = \alpha_1(1)\alpha_2(s) = a_1\alpha_2(s)$ and $\alpha_3(s) = \alpha_1(s)\alpha_2(1) = \alpha_1(s)a_2$ for $s \in S$. Hence, $\alpha_3(s) = a_1\alpha_2(s) = a_1a_1^{-1}(a_1\alpha_2(s)) = a_1a_1^{-1}(\alpha_1(s)a_2) = a_1(a_1^{-1}\alpha_1(s))a_2 = a_1\varphi(s)a_2$ for $s \in S$. Thus, $\alpha_1(s) = a_1\varphi(s)$, $\alpha_2(s) = \varphi(s)a_2$, $\alpha_3(s) = a_1\varphi(s)a_2$ for $s \in S$. It is easy to check that φ is a homomorphism.

The proof of the second part of this theorem is obvious.

Remark 1. Theorem 1 remains true if instead of the assumption that $\alpha_2(1) \in R(T)$, the element $\alpha_2(1)$ is supposed to be a right-cancellative element in the semigroup T .

Corollary 1. Let S be a groupoid with identity, and let T be a group. A triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (2) is a general solution of equation (1) if $\varphi: S \rightarrow T$ is an arbitrary homomorphism and $a_1, a_2 \in T$ are arbitrary elements.

Corollary 2. Let $(\alpha_1, \alpha_2, \alpha_3)$ be a homotopy from a groupoid S with identity into a group T . If α_i is a bijection for a certain $i \in \{1, 2, 3\}$, then $(\alpha_1, \alpha_2, \alpha_3)$ is an isotopy from S onto T .

Notice that formula (2) does not yield the general solution of equation (1). To show this, we consider the following

Example. Let $\{1, s\}$ be a group endowed with the operation:

	1	s
1	1	s
s	s	1

Let $T = \{1, t, 0\}$ be a monoid with the operation:

	1	t	0
1	1	t	0
t	t	1	0
0	0	0	0

Consider all the functions from the set T^S :

$$\begin{aligned}\alpha_1(1) &= 1 & \alpha_4(1) &= t & \alpha_7(1) &= 0 \\ \alpha_1(s) &= 1 & \alpha_4(s) &= 1 & \alpha_7(s) &= 1 \\ \alpha_2(1) &= 1 & \alpha_5(1) &= t & \alpha_8(1) &= 0 \\ \alpha_2(s) &= t & \alpha_5(s) &= t & \alpha_8(s) &= t \\ \alpha_3(1) &= 1 & \alpha_6(1) &= t & \alpha_9(1) &= 0 \\ \alpha_3(s) &= 0 & \alpha_6(s) &= 0 & \alpha_9(s) &= 0.\end{aligned}$$

The functions $\alpha_1, \alpha_2, \alpha_9$ are the only homomorphisms from S into T . The triple of functions $(\alpha_3, \alpha_9, \alpha_9)$ satisfies equation (1). It is easy to check that the function α_3 cannot be written in form (2) for any homomorphisms α_1, α_2 , and α_9 .

Consider on groupoids S and T the following functional equation

$$(3) \quad \alpha_3(\mu(s_1)\lambda(s_2)) = \alpha_1(s_1)\alpha_2(s_2)$$

for arbitrary $s_1, s_2 \in S$, where $\mu, \lambda: S \rightarrow S$ are given bijections and $\alpha_1, \alpha_2, \alpha_3: S \rightarrow T$ are unknown functions.

Notice that equation (3) is equivalent to the equation

$$(4) \quad \alpha_3(s_1 s_2) = (\alpha_1 \mu^{-1})(s_1)(\alpha_2 \lambda^{-1})(s_2)$$

for arbitrary $s_1, s_2 \in S$.

The following lemma is an immediate consequence of equation (4) and Theorem 1.

Lemma 1. *Let S be a groupoid with identity, and let T be a semigroup. If a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ is a solution of equation (3) on S and T such that $\alpha_1 \mu^{-1}(1)$ and $\alpha_2 \lambda^{-1}(1)$ are elements from $R(T)$, then there exists a homomorphism $\varphi: S \rightarrow T$ and there exist elements $a_1, a_2 \in R(T)$ such that*

$$(5) \quad \alpha_1 = L_{a_1} \varphi \mu, \quad \alpha_2 = R_{a_2} \varphi \lambda, \quad \alpha_3 = L_{a_1} R_{a_2} \varphi.$$

If the function $\varphi: S \rightarrow T$ is an arbitrary homomorphism and $a_1, a_2 \in T$ are arbitrary elements, then a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (5) is a solution of equation (3).

Corollary 3. *Let S be a groupoid with identity, and let T be a group. A triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ of form (5) is a general solution of equation (3) if $\varphi: S \rightarrow T$ is an arbitrary homomorphism and $a_1, a_2 \in T$ are arbitrary elements.*

Let X be an arbitrary non-empty set, and let S be an arbitrary groupoid. The set X^X endowed with the composition of functions is a monoid.

Consider the following functional equation

$$(6) \quad F_1(F_2(x, s_2), s_1) = F_3(x, s_1 s_2)$$

for arbitrary $x \in X$ and $s_1, s_2 \in S$, where $F_i: X \times S \rightarrow X$ for $i=1, 2, 3$ are unknown functions.

Lemma 2. *Let X be an arbitrary non-empty set, and let S be an arbitrary groupoid. A triple of functions (F_1, F_2, F_3) is a solution of equation (6) if and only if there exists a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ being a solution of equation (1) on the groupoid S and the monoid X^X , and $\alpha_i(s)(x) = F_i(x, s)$ ($i = 1, 2, 3$) for arbitrary $x \in X$ and $s \in S$.*

We omit the simple proof of this lemma.

We recall the following known facts.

A function $f \in X^X$ is an invertible element in the monoid X^X iff the function f is a bijection on the set X . A function $f \in X^X$ is a right-cancellative element in the monoid X^X iff the function f maps the set X onto the set X .

Theorem 2. *Let X be an arbitrary non-empty set, and let S be an arbitrary groupoid with identity. If a triple of functions (F_1, F_2, F_3) is a solution of equation (6) on S and the functions $F_1(x, 1)$ and $F_2(x, 1)$ are bijections on the set X , then there exists a homomorphism and there exist functions $f_1, f_2 \in X^X$ such that*

$$(7) \quad \begin{aligned} F_1(x, s) &= L_{f_1} \varphi(s)(x), & F_2(x, s) &= R_{f_2} \varphi(s)(x), \\ F_3(x, s) &= L_{f_1} R_{f_2} \varphi(s)(x) \end{aligned}$$

for arbitrary $x \in X$ and $s \in S$. If $\varphi: S \rightarrow X^X$ is an arbitrary homomorphism and $f_1, f_2 \in X^X$ are arbitrary functions, then a triple of functions (F_1, F_2, F_3) of form (7) is a solution of equation (6).

PROOF. Since the triple of functions (F_1, F_2, F_3) is a solution of equation (6), so according to Lemma 2, there exists a triple of functions $(\alpha_1, \alpha_2, \alpha_3)$ being a solution of equation (1) on the groupoid S and the monoid X^X such that

$$(8) \quad F_i(x, s) = \alpha_i(s)(x), \quad i = 1, 2, 3$$

for arbitrary $x \in X$ and $s \in S$.

Notice that $\alpha_i(1)(x) = F_i(x, 1)$ are bijections for $i = 1, 2$. Thus in virtue of Theorem 1, there exist functions $f_1, f_2 \in X^X$ and there exists a homomorphism $\varphi: S \rightarrow X^X$ such that $\alpha_1 = L_{f_1} \varphi$, $\alpha_2 = R_{f_2} \varphi$, $\alpha_3 = L_{f_1} R_{f_2} \varphi$. Hence and by (8) we get the functions F_i ($i = 1, 2, 3$) in form (7).

The proof of the first part of this theorem is completed. The easy proof of the second part is omitted.

Remark 2. Theorem 2 remains true if instead of the assumption that $F_2(x, 1)$ is a bijection, the function $F_2(x, 1)$ is supposed to map the set X onto the set X .

In the sequel we shall be concerned with the Pexider equation on n -semigroups. Definitions 2—5 and most of the notations are used according to papers [1] and [2].

Let $S(\circ)$ be a binary semigroup. The symbol s_m^n denotes either the sequence $s_m, s_{m+1}, \dots, s_n$ or the element $s_m \circ s_{m+1} \circ \dots \circ s_n$ for arbitrary $s_m, s_{m+1}, \dots, s_n \in S$ if $m \leq n$. The meaning of this symbol will uniquely result from the context. If $m > n$, then s_m^n is an empty symbol. The n -termed sequence $s, s, \dots, s$ is denoted by s^n and x^0 is an empty symbol.

Definition 1. A non-empty set S endowed with an n -ary operation ($n \geq 2$) is called an n -groupoid. n -groupoids will be written as $S(\)$ or $S[\]$.

Definition 2. An n -groupoid $S()$ is said to be an n -quasigroup if for an arbitrary $i \in \{1, \dots, n\}$ the equation

$$(s_1^{i-1}, x, s_{i+1}^n) = s$$

has the unique solution for arbitrary $s_1, \dots, s_n \in S$.

Definition 3. An n -groupoid $S()$ is said to be an n -semigroup if for arbitrary $i, j \in \{1, \dots, n\}$ the following equality is satisfied

$$(s_1^{i-1}, (s_i^{i+n-1}), s_{i+n}^{2n-1}) = (s_1^{j-1}, (s_j^{j+n-1}), s_{j+n}^{2n-1})$$

for arbitrary $s_1, \dots, s_{2n-1} \in S$.

An element $e \in S$ is said to be an identity of the n -groupoid $S()$ if $(e^{i-1}, s, e^{n-i}) = s$ for every $s \in S$ and for every $i \in \{1, \dots, n\}$.

If there exists an identity of an n -semigroup $S()$, then $S()$ is called an n -monoid. The identity of a binary monoid will be denoted by 1.

Definition 4. If an n -quasigroup $S()$ is an n -semigroup, then $S()$ is called an n -group.

Let S_i ($i=1, \dots, n$) be non-empty subsets of an n -groupoid $S()$. We define the following set:

$$(S_1, S_2, \dots, S_n) := \{(s_1, s_2, \dots, s_n) \in S : s_i \in S_i \text{ for } i = 1, \dots, n\}.$$

Definition 5. An element s of an n -groupoid $S()$ is called k -invertible if $(s^{k-1}, s, s^{n-k}) = s$. If an element $s \in S$ is 1-invertible and n -invertible in an n -groupoid $S()$, then s is called a bilateral invertible element. If for an arbitrary $k \in \{1, \dots, n\}$ an element $s \in S$ is k -invertible in an n -groupoid $S()$, then s is called an invertible element. The set of all invertible elements in an n -groupoid $S()$ is denoted by $R(S)$.

Theorem 3. (GLUSKIN [2]) Every bilateral invertible element of an n -semigroup is an invertible element.

Theorem 4. (GLUSKIN [2]) Let $S()$ be an arbitrary n -semigroup for which $R(S) \neq \emptyset$.

On the set S one can define the binary operation $\circ$ such that:

- 1) $(s_1^n) = s_1 \circ \lambda(s_2) \circ \lambda^2(s_3) \circ \dots \circ \lambda^{n-1}(s_n) \circ a$ for arbitrary $s_1, s_2, s_3, \dots, s_n \in S$;
- 2) $S(\circ)$ is a binary monoid with the same set $R(S)$ of invertible elements;
- 3) λ is an automorphism of the monoid $S(\circ)$;
- 4) $a \in R(S)$ and $\lambda(a) = a$;
- 5) $\lambda^{n-1}(x) = a \circ x \circ a^{-1}$ for every $x \in S$.

The binary monoid $S(\circ)$ will be called a monoid associated with the n -semigroup $S()$ and it will be denoted by $(S, \circ, \lambda, a)$. A monoid associated with an n -group is a binary group. A monoid associated with a binary monoid $S(\cdot)$ is the same binary monoid $S(\cdot)$.

Corollary 4 (GLUSKIN [2]) Let $S()$ be an n -monoid. On the set S one can define the binary operation $\circ$ such that:

- 1) $(s_1^n) = s_1 \circ s_2 \circ \dots \circ s_n$ for arbitrary $s_1, s_2, \dots, s_n \in S$;
- 2) $S(\circ)$ is a binary monoid.

For simplicity, the monoid associated with the n -monoid $S(\cdot)$ will be denoted by $S(\circ)$ or S .

Definition 6. A Pexider equation on n -groupoids $S(\cdot)$ and $T[\cdot]$ is the functional equation

$$(9) \quad \alpha_{n+1}((s_1, s_2, \dots, s_n)) = [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)]$$

for arbitrary $s_1, s_2, \dots, s_n \in S$, where $\alpha_i: S \rightarrow T$ (for $i=1, \dots, n+1$) are unknown functions.

A sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ being a solution of equation (9) is called a homotopy from an n -groupoid $S(\cdot)$ into an n -groupoid $T[\cdot]$.

If $\alpha_1, \dots, \alpha_{n+1}$ are bijections, then a homotopy $(\alpha_1, \dots, \alpha_{n+1})$ is called an isotopy from an n -groupoid $S(\cdot)$ onto an n -groupoid $T[\cdot]$.

If a sequence $(\alpha, \alpha, \dots, \alpha)$ is a homotopy (an isotopy) from an n -groupoid $S(\cdot)$ into (onto) an n -groupoid $T[\cdot]$, then the function α is called a homomorphism (an isomorphism) from an n -groupoid $S(\cdot)$ into (onto) an n -groupoid $T[\cdot]$.

Theorem 5. *If two n -monoids are isotopic, then they are isomorphic.*

PROOF. Let $(\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ be an isotopy from an n -monoid $S(\cdot)$ onto an n -monoid $T[\cdot]$, and let $S(\circ)$ and $T[\cdot]$ be monoids associated with $S(\cdot)$ and $T[\cdot]$ respectively. By the definition of the isotopy we have $\alpha_{n+1}((s_1, s_2, \dots, s_n)) = [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)]$ for arbitrary $s_1, s_2, \dots, s_n \in S$. In virtue of Corollary 4 we obtain (10) $\alpha_{n+1}(s_1 \circ s_2 \circ \dots \circ s_n) = \alpha_1(s_1) \cdot \alpha_2(s_2) \cdot \dots \cdot \alpha_n(s_n)$ for arbitrary $s_1, s_2, \dots, s_n \in S$. Let us put $a_i := \alpha_i(1)$ for $i=1, \dots, n$. By (10) we have $\alpha_{n+1}(s_1 \circ s_2) = \alpha_1(s_1) \cdot (\alpha_2(s_2) \cdot a_3^n)$ for arbitrary $s_1, s_2 \in S$. Put $\beta(s_2) := \alpha_2(s_2) \cdot a_3^n$ for every $s_2 \in S$, and so $\beta = R_{a_3^n} \alpha_2$. Besides, $\alpha_{n+1}(s_2) = a_1 \cdot \alpha_2(s_2) \cdot a_3^n$ for every $s_2 \in S$, i.e. $\alpha_{n+1} = L_{a_1} R_{a_3^n} \alpha_2$. Since α_2 and α_{n+1} are bijections and $L_{a_1} R_{a_3^n} = R_{a_3^n} L_{a_1}$ so L_{a_1} and $R_{a_3^n}$ are bijections. Thus, the function β is a bijection. Whence $\alpha_{n+1}(s_1 \circ s_2) = \alpha_1(s_1) \cdot \beta(s_2)$ for arbitrary $s_1, s_2 \in S$, and so the monoids $S(\circ)$ and $T[\cdot]$ are isotopic. It is the well known fact that isotopic binary monoids are isomorphic (cf. [4]). Thus there exists an isomorphism $\varphi: S \rightarrow T$ of the monoids $S(\circ)$ and $T[\cdot]$. It is easy to see that φ is also an isomorphism of the n -monoids $S(\cdot)$ and $T[\cdot]$.

Remark 3. Let us notice that two isotopic n -groups ($n > 2$) need not be isomorphic (a suitable example can be found in Belousov [1]).

It is easy to prove the following two propositions.

Proposition 1. *Every n -groupoid isotopic to an n -quasigroup is an n -quasigroup.*

Proposition 2. *Every n -semigroup isotopic to an n -quasigroup is an n -group.*

An n -groupoid isotopic to an n -group need not be an n -group (a suitable example can be found in the paper [4], p. 101).

Theorem 6. *Let $S(\cdot)$ and $T[\cdot]$ be n -semigroups for which $R(S) \neq \emptyset$ and $R(T) \neq \emptyset$. Let $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ be monoids associated with $S(\cdot)$ and $T[\cdot]$, respectively. If a sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ is a solution of equation (9) on the n -semigroups $S(\cdot)$ and $T[\cdot]$, and $\alpha_i(1)$ (for $i=1, \dots, n-1$), $\alpha_n(a^{-1})$ are elements from*

$R(T)$, then there exists a homomorphism $\varphi: S \rightarrow T$ of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$, and there exist elements $a_1, \dots, a_n \in R(T)$ such that

$$(11) \quad \begin{cases} \alpha_1 = L_{a_1} \varphi, \\ \alpha_k = \mu^{1-k} L_{a_2^{k-1}}^{-1} R_{a_2^k} \varphi \lambda^{k-1} \text{ for } k = 2, \dots, n-1, \\ \alpha_n = \mu^{1-n} L_{a_2^{n-1}}^{-1} R_{a_2^n} \varphi \lambda^{n-1}, \\ \alpha_{n+1} = L_{a_1} R_{a_2^n} \cdot b \varphi. \end{cases}$$

If $\varphi: S \rightarrow T$ is an arbitrary homomorphism of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ and $a_2, \dots, a_{n-1} \in R(T)$ while a_1, a_n are arbitrary elements from T , then a sequence of functions of form (11) is a solution of equation (9) on n -semigroups $A(\)$ and $T[\]$.

PROOF. If $n=2$ then this theorem is an immediate consequence of Theorem 1. Suppose that $n \geq 3$. It follows from the assumptions of this theorem that $\alpha_{n+1}((s_1, s_2, \dots, s_n)) = [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)]$ for arbitrary $s_1, s_2, \dots, s_n \in S$. In virtue of Theorem 4 we have $\alpha_{n+1}(s_1 \circ \lambda(s_2) \circ \dots \circ \lambda^{n-1}(s_n) \circ a) = \alpha_1(s_1) \cdot \mu \alpha_2(s_2) \cdot \dots \cdot \mu^{n-1} \alpha_n(s_n) \cdot b$. Put $a_i := \mu^{i-1} \alpha_i(1)$ (for $i=1, \dots, n-1$) and $a_n := \mu^{n-1} \alpha_n(a^{-1})$. Whence $\alpha_{n+1}(s_1 \circ \lambda(s_2)) = \alpha_1(s_1) \cdot (\mu \alpha_2(s_2) \cdot a_3^n \cdot b)$ for arbitrary $s_1, s_2 \in S$. Putting $\beta_2 := R_{a_3^n} \cdot b \mu \alpha_2$ we have $\alpha_{n+1}(s_1 \circ \lambda(s_2)) = \alpha_1(s_1) \cdot \beta_2(s_2)$ for arbitrary $s_1, s_2 \in S$. Since $\alpha_1(1) = a_1$ and $\beta_2 \lambda^{-1}(1) = \beta_2(1) = \mu \alpha_2(1) \cdot a_3^n \cdot b = a_2^n \cdot b$ are elements from $R(T)$, then according to Lemma 1 there exists a homomorphism $\varphi: S \rightarrow T$ of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ such that $\alpha_1 = L_{a_1} \varphi$, $\beta_2 = R_{a_2^n} \cdot b \varphi \lambda$, $\alpha_{n+1} = L_{a_1} R_{a_2^n} \cdot b \varphi$. Hence, $R_{a_3^n} \cdot b \mu \alpha_2 = R_{a_2^n} \cdot b \varphi \lambda$, and so $\alpha_2 = \mu^{-1} R_{a_2} \varphi \lambda$. Thus, $\alpha_1 = L_{a_1} \varphi$, $\alpha_2 = \mu^{-1} R_{a_2} \varphi \lambda$, $\alpha_{n+1} = L_{a_1} R_{a_2^n} \cdot b \varphi$.

Let us assume that $3 \leq k \leq n-1$. Putting $s_1=1, \dots, s_{k-2}=1, s_{k+1}=1, \dots, s_{n-1}=1, s_n=a^{-1}$ we have $\alpha_{n+1}(\lambda^{k-2}(s_{k-1}) \circ \lambda^{k-1}(s_k)) = (a_1^{k-2} \cdot \mu^{k-2} \alpha_{k-1}(s_{k-1})) \cdot (\mu^{k-1} \alpha_k(s_k) \cdot a_{k+1}^n \cdot b)$ for arbitrary $s_{k-1}, s_k \in S$. Let us put $\beta_{k-1} := L_{a_1^{k-2}} \mu^{k-2} \alpha_{k-1}$, $\beta_k := R_{a_{k+1}^n} \cdot b \mu^{k-1} \alpha_k$ and so $\alpha_{n+1}(\lambda^{k-2}(s_{k-1}) \circ \lambda^{k-1}(s_k)) = \beta_{k-1}(s_{k-1}) \cdot \beta_k(s_k)$ for arbitrary $s_{k-1}, s_k \in S$. Let us notice that $\beta_{k-1} \lambda^{2-k}(1) = \beta_{k-1}(1) = a_1^{k-2} \cdot \mu^{k-2} \alpha_{k-1}(1) = a_1^{k-1}$ and $\beta_k \lambda^{1-k}(1) = \beta_k(1) = \mu^{k-1} \alpha_k(1) \cdot a_{k+1}^n \cdot b = a_k^n \cdot b$ are elements from $R(T)$. It follows from Lemma 1 that there exists a homomorphism $\psi: S \rightarrow T$ of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ such that $\beta_{k-1} = L_{a_1^{k-1}} \psi \lambda^{k-2}$, $\beta_k = R_{a_k^n} \cdot b \psi \lambda^{k-1}$, $\alpha_{n+1} = L_{a_1^{k-1}} R_{a_k^n} \cdot b \psi$. Thus, we obtain $L_{a_1^{k-1}} R_{a_k^n} \cdot b \psi = L_{a_1} R_{a_2^n} \cdot b \varphi$, whence $L_{a_2^{k-1}} \psi = R_{a_2^{k-1}} \varphi$, and so $\psi = L_{a_2^{k-1}}^{-1} R_{a_2^{k-1}} \varphi$. Since $R_{a_{k+1}^n} \cdot b \mu^{k-1} \alpha_k = R_{a_k^n} \cdot b \psi \lambda^{k-1}$, then $\alpha_k = \mu^{1-k} R_{a_k} \psi \lambda^{k-1} = \mu^{1-k} R_{a_k} L_{a_2^{k-1}}^{-1} R_{a_2^{k-1}} \varphi \lambda^{k-1} = \mu^{1-k} L_{a_2^{k-1}}^{-1} R_{a_2^k} \varphi \lambda^{k-1}$ for $k=3, \dots, n-1$. Let us notice that the function α_2 can also be written in the above form.

To determine the function α_n put $s_1=1, \dots, s_{n-2}=1$ and consider the following equation $\alpha_{n+2}(\lambda^{n-2}(s_{n-1}) \circ \lambda^{n-1}(s_n) \circ a) = (a_1^{n-2} \mu^{n-2} \alpha_{n-1}(s_{n-1})) \cdot (\mu^{n-1} \alpha_n(s_n) \cdot b)$ for arbitrary $s_{n-1}, s_n \in S$. Set $\beta_{n-1} := L_{a_1^{n-2}} \mu^{n-2} \alpha_{n-1}$ and $\beta_n := R_b \mu^{n-1} \alpha_n$. Whence, $\alpha_{n+1}(\lambda^{n-2}(s_{n-1}) \circ R_a \lambda^{n-1}(s_n)) = \beta_{n-1}(s_{n-1}) \cdot \beta_n(s_n)$ for arbitrary $s_{n-1}, s_n \in S$. Notice that $\beta_{n-1} \lambda^{2-n}(1) = \beta_{n-1}(1) = a_1^{n-2} \cdot \mu^{n-2} \alpha_{n-1}(1) = a_1^{n-1}$ and $\beta_n (R_a \lambda^{n-1})^{-1}(1) = \beta_n \lambda^{1-n} R_{a^{-1}}(1) = \beta_n \lambda^{1-n}(a^{-1}) = \beta_n(a^{-1}) = \mu^{n-1} \alpha_n(a^{-1}) \cdot b = a_n \cdot b$. According to Lem-

ma 1 there exists a homomorphism $\chi: S \rightarrow T$ of monoids $(S, \circ, \lambda, a)$ and $(T, \cdot, \mu, b)$ such that $\beta_{n-1} = L_{a_1^{n-1}} \chi \lambda^{n-2}$, $\beta_n = R_{a_n \cdot b} \chi R_a \lambda^{n-1}$, $\alpha_{n+1} = L_{a_1^{n-1}} R_{a_n \cdot b} \chi$. Hence, $L_{a_1^{n-1}} R_{a_n \cdot b} \chi = L_{a_1} R_{a_2^n \cdot b} \varphi$, and so $\chi = L_{a_2^{n-1}}^{-1} R_{a_2^n \cdot b} \varphi$. Thus, $R_b \mu^{n-1} \alpha_n = R_{a_n \cdot b} \chi R_a \lambda^{n-1} = R_{a_n \cdot b} L_{a_2^{n-1}}^{-1} R_{a_2^n \cdot b} \varphi R_a \lambda^{n-1}$, and so $\alpha_n = \mu^{1-n} L_{a_2^{n-1}}^{-1} R_{a_2^n} \varphi R_a \lambda^{n-1}$. Finally, we obtain the formulas of form (11).

We shall prove the second part of this theorem. Let us notice that

$$\begin{aligned} \alpha_{n+1}((s_1, s_2, s_3, \dots, s_n)) &= \alpha_{n+1}(s_1 \circ \lambda(s_2) \circ \lambda^2(s_3) \circ \dots \circ \lambda^{n-1}(s_n) \circ a) = \\ &= L_{a_1} R_{a_2^n \cdot b} \varphi(s_1 \circ \lambda(s_2) \circ \lambda^2(s_3) \circ \dots \circ \lambda^{n-1}(s_n) \circ a) = \\ &= a_1 \cdot \varphi(s_1) \cdot \varphi \lambda(s_2) \cdot \varphi \lambda^2(s_3) \cdot \dots \cdot \varphi \lambda^{n-1}(s_n) \cdot \varphi(a) \cdot a_2^n \cdot b = \\ &= (a_1 \cdot \varphi(s_1)) \cdot (\varphi \lambda(s_2) \cdot a_2) \cdot ((a_2)^{-1} \cdot \varphi \lambda^2(s_3) \cdot a_2^3) \cdot \dots \cdot ((a_2^{n-1})^{-1} \cdot \\ &\cdot \varphi \lambda^{n-1}(s_n) \cdot \varphi(a) \cdot a_2^n \cdot b) = \alpha_1(s_1) \cdot \mu \alpha_2(s_2) \cdot \mu^2 \alpha_3(s_3) \cdot \dots \cdot \mu^{n-1} \alpha_n(s_n) \cdot b = \\ &= [\alpha_1(s_1), \alpha_2(s_2), \alpha_3(s_3), \dots, \alpha_n(s_n)] \end{aligned}$$

for arbitrary $s_1, \dots, s_n \in S$.

The proof of this theorem is completed.

Corollary 5. Let $S(\)$ be an n -semigroup such that $R(S) \neq \emptyset$, and let $(S, \circ, \lambda, a)$ be a monoid associated with $S(\)$. Let $T[\]$ be an n -group, and let $(T, \cdot, \mu, b)$ be a group associated with $T[\]$. A sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ of form (11) is a general solution of equation (9) if $\varphi: S \rightarrow T$ is an arbitrary homomorphism from the monoid $(S, \circ, \lambda, a)$ into the group $(T, \cdot, \mu, b)$ and $a_1, \dots, a_n \in T$ are arbitrary elements.

Corollary 6. Let $S(\)$ be an n -semigroup such that $R(S) \neq \emptyset$, and let $T[\]$ be an n -group. Let $(\alpha_1, \dots, \alpha_{n+1})$ be a homotopy from the n -semigroup $S(\)$ into the n -group $T[\]$. If α_i is a bijection for a certain $i \in \{1, \dots, n+1\}$, then $(\alpha_1, \dots, \alpha_{n+1})$ is an isotopy. Furthermore, the monoid $(S, \circ, \lambda, a)$ is isomorphic to the group $(T, \cdot, \mu, b)$.

Let X be an arbitrary non-empty set, and let $S(\)$ be an arbitrary n -groupoid ($n \geq 3$).

Let us consider the following functional equation

$$(12) \quad F_1(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1) = F_{n+1}(x, (s_1, s_2, \dots, s_n))$$

for arbitrary $x \in X$ and $s_1, s_2, \dots, s_n \in S$, where $F_i: X \times S \rightarrow X$ (for $i = 1, \dots, n+1$) are unknown functions. Equation (12) is an analogue for n -groupoids of the functional equation considered by Grzaślewicz (cf. [3]).

We introduce an n -ary operation $[\]$ on X^X by defining

$$[f_1, f_2, \dots, f_n] := f_1 f_2 \dots f_n$$

for arbitrary functions $f_1, f_2, \dots, f_n \in X^X$.

The expression on the right-side of the above equality is the n -fold composition of functions.

Then $X^X[\]$ is an n -monoid.

Theorem 7. Let X be an arbitrary non-empty set, and let $S(\)$ be an arbitrary n -groupoid. A sequence of functions $(F_1, \dots, F_{n+1})$ is a solution of equation (12) on the n -groupoid $S(\)$ if and only if there exists a sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ being

a solution of equation (9) on the n -groupoid $S(\)$ and the n -monoid $X^X[\]$, and $F_i(x, s) = \alpha_i(s)(x)$ for arbitrary $x \in X$, $s \in S$, $i \in \{1, \dots, n+1\}$.

PROOF. (i) Let us assume that the sequence $(F_1, F_2, \dots, F_{n+1})$ is a solution of equation (12). Then,

$$\begin{aligned} \alpha_{n+1}((s_1, s_2, \dots, s_n))(x) &= F_{n+1}(x, (s_1, s_2, \dots, s_n)) = \\ &= F_1(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1) = \alpha_1(s_1)(\alpha_2(s_2)(\dots(\alpha_n(s_n)(x))\dots)) = \\ &= (\alpha_1(s_1)\alpha_2(s_2)\dots\alpha_n(s_n))(x) = [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)](x) \end{aligned}$$

for arbitrary $x \in X$ and $s_1, s_2, \dots, s_n \in S$.

(ii) Let the sequence $(\alpha_1, \alpha_2, \dots, \alpha_{n+1})$ be a solution of equation (9) on the n -groupoid $S(\)$ and the n -monoid $X^X[\]$. Then, $F_1(F_2(\dots(F_n(x, s_n), \dots), s_2), s_1) = (\alpha_1(s_1)\alpha_2(s_2)\dots\alpha_n(s_n))(x) = [\alpha_1(s_1), \alpha_2(s_2), \dots, \alpha_n(s_n)](x) = \alpha_{n+1}((s_1, s_2, \dots, s_n))(x) = (F_{n+1}x, (s_1, s_2, \dots, s_n))$ for arbitrary $x \in X$ and $s_1, s_2, \dots, s_n \in S$.

Theorem 8. Let X be an arbitrary non-empty set. Let $S(\)$ be an n -semigroup such that $R(S) \neq \emptyset$, and let $(S, \circ, \lambda, a)$ be a monoid associated with $S(\)$. If a sequence of functions $(F_1, \dots, F_{n+1})$ is a solution of equation (12) on the n -semigroup $S(\)$ and the functions $F_i(x, 1)$ (for $i=1, \dots, n-1$), $F_n(x, a^{-1})$ are bijections on the set X , then there exists a homomorphism $\varphi: S \rightarrow X^X$ of binary monoids $(S, \circ, \lambda, a)$ and X^X , and there exist functions $f_1, \dots, f_n \in X^X$ such that

$$(13) \quad \begin{cases} F_1(x, s) = L_{f_1}\varphi(s)(x), \\ F_k(x, s) = L_{f_2^{k-1}}R_{f_2^k}\varphi\lambda^{k-1}(s)(x) \quad \text{for } k = 2, \dots, n-1, \\ F_n(x, s) = L_{f_2^{n-1}}R_{f_2^n}\varphi R_a\lambda^{n-1}(s)(x), \\ F_{n+1}(x, s) = L_{f_1}R_{f_2^n}\varphi(s)(x), \end{cases}$$

for arbitrary $x \in X$ and $s \in S$.

If $\varphi: S \rightarrow X^X$ is an arbitrary homomorphism of the binary monoids $(S, \circ, \lambda, a)$ and X^X , and $f_2, \dots, f_{n-1}$ are arbitrary bijections on the set X while $f_1, f_n \in X^X$ are arbitrary functions, then a sequence of functions $(F_1, \dots, F_{n+1})$ of form (13) is a solution of equation (12) on the n -semigroup $S(\)$.

PROOF. Let the sequence of functions $(F_1, \dots, F_{n+1})$ be a solution of equation (12) satisfying the assumptions of this theorem. It follows from Theorem 7 that there exists a sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ being a solution of equation (9) on the n -semigroup $S(\)$ and the n -monoid $X^X[\]$, and furthermore $F_i(x, s) = \alpha_i(s)(x)$ for arbitrary $x \in X$, $s \in S$, $i \in \{1, \dots, n+1\}$. The functions $f_i(x) := \alpha_i(1)(x) = F_i(x, 1)$ (for $i=1, \dots, n-1$) and $f_n(x) := \alpha_n(a^{-1})(x) = F_n(x, a^{-1})$ for $x \in X$ are bijections on the set X . It follows from Theorem 6 that there exists a homomorphism $\varphi: S \rightarrow X^X$ of monoids $(S, \circ, \lambda, a)$ and X^X such that

$$(14) \quad \begin{aligned} \alpha_1 &= L_{f_1}\varphi, \\ \alpha_k &= L_{f_2^{k-1}}R_{f_2^k}\varphi\lambda^{k-1} \quad \text{for } k = 2, \dots, n-1, \\ \alpha_n &= L_{f_2^{n-1}}R_{f_2^n}\varphi R_a\lambda^{n-1}, \\ \alpha_{n+1} &= L_{f_1}R_{f_2^n}\varphi. \end{aligned}$$

Thus, applying the equalities $F_i(x, s) = \alpha_i(s)(x)$ we obtain formulas (13).

We shall prove the second part of the theorem. It follows from Theorem 6 that the sequence of functions $(\alpha_1, \dots, \alpha_{n+1})$ of form (14) is a solution of equation (9) on the n -semigroup $S(\)$ and the n -monoid $X^X[\]$. Notice that for the functions F_i of form (13) we have $F_i(x, s) = \alpha_i(s)(x)$ for arbitrary $x \in X$, $s \in S$, $i \in \{1, \dots, n+1\}$. Thus according to Theorem 7, the sequence of functions $(F_1, \dots, F_{n+1})$ of form (13) is a solution of equation (12) on the n -semigroup $S(\)$.

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(Received November 25, 1986)