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Stability of the Cauchy equation with variable bound

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Abstract. We investigate the problem of stability of the Cauchy equation on $\mathbb{R}_+$. As a result we obtain a positive answer to the problem of G. Maksa posed on *34 ISFE* concerning the Hyers-Ulam stability of the equation

$$f(xy) = xf(y) + yf(x)$$

on the unit interval.

On the *34-th International Symposium on Functional Equations* G. Maksa posed two problems ([3]).

Problem A. Let $\phi :]0,1] \to \mathbb{R}$ and $0 < \varepsilon \in \mathbb{R}$. Suppose that

$$|\phi(xy) - x\phi(y) - y\phi(x)| \le \varepsilon, \qquad x, y \in]0,1]. \tag{1}$$

Does there exist $a :]0,1] \to \mathbb{R}$ such that

$$a(xy) - xa(y) - ya(x) = 0, \quad x, y \in]0,1]$$

and $\phi - a$ is bounded?

Problem B. Find all functions $f, g : \mathbb{R}_+ := [0, \infty[\to \mathbb{R}$ satisfying the functional inequality

$$|f(u+v) - f(u) - f(v)| \le g(u+v), \quad u \ge 0, v \ge 0.$$

As the *Problem B* is very general, we propose to investigate a more specific one:

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Problem 1. Let E be a Banach space, let G be a commutative semigroup, and let $g : G \to \mathbb{R}_+$ be a given function. Does there exist $K > 0$ such that for each $f : G \to E$ satisfying

$$\|f(x+y) - f(x) - f(y)\| \leq g(x+y) \quad \text{ for } x, y \in G$$

there exists an additive $A : G \to E$ satisfying the inequality

$$\|f(x) - A(x)\| \leq Kg(x) \quad \text{ for } x \in G\,?$$

If G is a group then this problem has a positive solution for all functions g in a large class of Banach spaces (cf. [2]). However, for $G = \mathbb{R}_+$ this statement fails to hold – Z. Gajda presents in [1] an example of a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $|f(x+y) - f(x) - f(y)| \leq x + y$ for $x, y \in \mathbb{R}_+$, but there exist no additive mapping $A : \mathbb{R}_+ \to \mathbb{R}$ and $K > 0$ such that $|f(x) - A(x)| \leq Kx$ for $x \in \mathbb{R}_+$.

In this paper we show that if $G = \mathbb{R}_+$ and the function g is, in some sense, fast increasing, then the answer turns out to be positive. Thus we give a partial answer to *Problem B*. This enables us to positively solve *Problem A*.

Now we introduce some notations. In the whole paper we assume that E is a sequentially complete topological vector space and that V is a closed, convex, bounded subset of E symmetric with respect to zero. For $f : \mathbb{R}_+ \to E$ we denote the Cauchy difference of f by

$$\mathcal{C}f(x,y) := f(x+y) - f(x) - f(y).$$

For $h \in]0, \infty[$ and $x \in \mathbb{R}$ we define

$$E_h[x] := \max\{n \in \mathbb{Z} : nh < x\},\ F_h[x] := x - E_h[x]h.$$

Clearly $E_h[x] \in \mathbb{Z}$, $F_h[x] \in]0, h]$. To avoid distinguishing several cases and to shorten some considerations we will use the following convention: if $m, n \in \mathbb{N}$, $m > n$ then by $\sum\limits_{i=m}^{n} a_i$ we mean zero.

In our investigations we will need the following proposition (some connected results can be found in [5]).

Proposition 1. *Let $h \in\,]0,\infty[\,$, and let $f : [0,h] \to E$ be a function such that*

$$\mathcal{C}f(x,y) \in V \quad \text{for } x,y,x+y \in [0,h]. \tag{2}$$

Then there exists a unique additive function $A : \mathbb{R}_+ \to E$ such that $A(h) = f(h)$ and

$$f(x) - A(x) \in 2V \quad \text{for } x \in [0,h].$$

Proof. We define $\widetilde{f} : \mathbb{R}_+ \to E$ by

$$\widetilde{f}(x) := E_h[x]f(h) + f(F_h[x]) \quad \text{for } x \in \mathbb{R}_+. \tag{3}$$

Clearly

$$\widetilde{f}|_{]0,h]} = f. \tag{4}$$

We show that $\mathcal{C}\widetilde{f}(x,y) \in 2V$ for $x,y \in \mathbb{R}_+$. If $F_h[x] + F_h[y] \in\,]0,h]$ then by (3),(4) and (2)

$$\mathcal{C}\widetilde{f}(x,y) = \mathcal{C}f(F_h[x],F_h[y]) \in V.$$

Now suppose that $F_h[x] + F_h[y] \in\,]h,2h[\,$. Since the Cauchy difference is symmetric we may assume without loss of generality that $F_h[x] \in\,]0,h[\,$. Then

$$\begin{aligned}\mathcal{C}\widetilde{f}(x,y) &= f(F_h[x]+F_h[y]-h) + f(h) - f(F_h[x]) - f(F_h[y])\\ &= \mathcal{C}f(F_h[x],h-F_h[x]) - \mathcal{C}f(F_h[x]+F_h[y]-h,h-F_h[x])\\ &\in V - V = 2V.\end{aligned}$$

If $F_h[x] + F_h[y] = 2h$, then $\mathcal{C}\widetilde{f}(x,y) = 0 \in 2V$.

Now by the generalized Hyers Theorem (cf. Th. 4,5, [4]) we obtain that there exists an additive function $A : \mathbb{R}_+ \to E$ such that

$$\widetilde{f}(x) - A(x) \in 2V \quad \text{for } x \in \mathbb{R}_+.$$

As $f(0) = -\mathcal{C}f(0,0) \in V$, this and (4) imply that

$$f(x) - A(x) \in 2V \quad \text{for } x \in [0,h].$$

Moreover, as $\widetilde{f}(nh) = nf(h)$, we have

$$\widetilde{f}(h) - A(h) = \frac{\widetilde{f}(nh) - A(nh)}{n} \in \frac{2}{n}V \quad \text{for } n \in \mathbb{N},$$

so $f(h) = \widetilde{f}(h) = A(h)$.

We show that A is unique. Suppose that there exists an additive A' with the same properties as A. Then

$$(A - A')(h) = 0,$$

which means that $A - A'$ is periodic with period h. As $A - A'$ is bounded on $[0, h]$, this implies that it is globally bounded, so it is the zero function. □

The following theorem is a partial answer to *Problem B*.

Theorem 1. *Let $h \in]0, \infty[$, and let $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be an arbitrary function such that $\sup\limits_{y,z,y+z \in [0,h]} g(y, z) < \infty$. Let $f : \mathbb{R}_+ \to E$ satisfy the condition*

$$\mathcal{C}f(x, y) \in g(x, y)V \quad \text{for } x, y \in \mathbb{R}_+.$$

Then there exists a unique additive function $A : \mathbb{R}_+ \to E$ such that $A(h) = f(h)$ and

$$f(x) - A(x) \in \Big(2 \sup_{y,z,y+z\in[0,h]} g(y, z) + \sum_{i=1}^{E_h[x]} g(x - ih, h)\Big)V, \tag{5}$$

$$f(x) - A(x) \in \Big(2^{n(x)+1} \sup_{y,z,y+z\in[0,h]} g(y, z) + \sum_{i=1}^{n(x)} 2^{i-1} g\Big(\frac{x}{2^i}, \frac{x}{2^i}\Big)\Big)V \tag{6}$$

for $x \in \mathbb{R}_+$, where $n(x)$ is the smallest nonnegative integer such that $\frac{x}{2^{n(x)}} \in [0, h]$.

Proof. By Proposition 1 there exists a unique additive $A : \mathbb{R}_+ \to E$ such that $A(h) = f(h)$ and

$$f(x) - A(x) \in 2 \sup_{y,z,y+z\in[0,h]} g(y, z)V \quad \text{for } x \in [0, h]. \tag{7}$$

At first we prove that A satisfies (5). For $x \in [0,h]$ this is trivial. Suppose that $x > h$. Then

$$f(x) - f(F_h[x]) - E_h[x]f(h) = \sum_{i=1}^{E_h[x]} \left(f(x-(i-1)h) - f(x-ih) - f(h)\right)$$
$$\in \sum_{i=1}^{E_h[x]} g(x-ih,h)V.$$

But $F_h[x] \in [0,h]$, so by (7)

$$\begin{aligned} f(x) - A(x) &= E_h[x](f(h) - A(h)) + (f(F_h[x]) - A(F_h[x]) \\ &\quad + (f(x) - f(F_h[x]h) - E_h[x]f(h)) \\ &\in \Big\{2 \sup_{y,z,y+z} g(y,z) + \sum_{i=1}^{E_h[x]} g(x-ih,h)\Big\}V. \end{aligned}$$

Now we show that A satisfies (6). For $x \in [0,h]$ this is obvious. Suppose that $x > h$. Then

$$\begin{aligned} f(x) - 2^{n(x)} f(\frac{x}{2^{n(x)}}) &= \sum_{i=1}^{n(x)} \Big(2^{i-1} f(\frac{x}{2^{i-1}}) - 2^i f\Big(\frac{x}{2^i}\Big)\Big) \\ &= \sum_{i=1}^{n(x)} 2^{i-1} \mathcal{C}f\Big(\frac{x}{2^i},\frac{x}{2^i}\Big) \\ &\in \sum_{i=1}^{n(x)} 2^{i-1} g\Big(\frac{x}{2^i},\frac{x}{2^i}\Big)V. \end{aligned}$$

However, $\dfrac{x}{2^{n(x)}} \in [0,h]$, which yields by (7) that

$$\begin{aligned} f(x) - A(x) &= \Big(f(x) - 2^{n(x)} f\Big(\frac{x}{2^{n(x)}}\Big)\Big) \\ &\quad - \Big(2^{n(x)} A\Big(\frac{x}{2^{n(x)}}\Big) - 2^{n(x)} f\Big(\frac{x}{2^{n(x)}}\Big)\Big) \\ &\in \Big(2^{n(x)+1} \sup_{y,z,y+z\in[0,h]} g(y,z) + \sum_{i=1}^{n(x)} 2^{i-1} g\Big(\frac{x}{2^i},\frac{x}{2^i}\Big)\Big)V. \end{aligned}$$

$\square$

Definition 1. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ will be called exponentially increasing if it is increasing and there exist $\gamma > 1$ and $h \in \mathbb{R}_+$ such that

$$g(x+h) \geq \gamma g(x) \quad \text{for } x \in \mathbb{R}_+.$$

Definition 2. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ will be called powerly increasing if it is increasing and there exists $\gamma > 1$ and $h \in \mathbb{R}_+$ such that

$$g(2x) \geq 2\gamma g(x) \quad \text{for } x \geq h.$$

Obviously every exponential increasing function is exponentially increasing and every power function of degree greater than one is powerly increasing.

Now we solve Problem 1 in the class of powerly increasing and in that of exponentially increasing functions.

Theorem 2. (i) *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is exponentially increasing with constants γ and h as in Definition 1, and that $g(0) > 0$.*

Let $K := 2\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma-1}$, and let $f : \mathbb{R}_+ \to E$ be an arbitrary function such that

$$\mathcal{C}f(x,y) \in g(x+y)V \quad \text{for } x, y \in \mathbb{R}_+.$$

Then there exists a unique additive function $A : \mathbb{R}_+ \to E$ such that $A(h) = f(h)$ and that

$$f(x) - A(x) \in Kg(x)V \quad \text{for } x \in \mathbb{R}_+.$$

(ii) *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}_+$ is powerly increasing with constants γ and h as in Definition 2, and that $g(0) > 0$.*

Let $K := 4\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma-1}$, and let $f : \mathbb{R}_+ \to E$ be an arbitrary function such that

$$\mathcal{C}f(x,y) \in g(x+y)V \quad \text{for } x, y \in \mathbb{R}_+.$$

Then there exists a unique additive function $A : \mathbb{R}_+ \to E$ such that $A(h) = f(h)$ and that

$$f(x) - A(x) \in Kg(x)V \quad \text{for } x \in \mathbb{R}_+.$$

PROOF. By Theorem 1 there exists a unique additive $A : \mathbb{R}_+ \to E$ such that $A(h) = f(h)$ and

$$f(x) - A(x) \in \Big(2 \sup_{y,z,y+z\in[0,h]} g(y+z) + \sum_{i=1}^{E_h[x]} g(x - ih + h)\Big)V, \tag{8}$$

$$f(x) - A(x) \in \Big(2^{n(x)+1} \sup_{y,z,y+z\in[0,h]} g(y+z) + \sum_{i=1}^{n(x)} 2^{i-1} g\Big(\frac{x}{2^i} + \frac{x}{2^i}\Big)\Big)V \tag{9}$$

for $x \in \mathbb{R}_+$, where $n(x)$ is as in Theorem 1.

(i) Obviously

$$2 \sup_{y,z,y+z\in[0,h]} g(y+z) = 2g(h) \le 2\frac{g(h)}{g(0)} g(x) \quad \text{for } x \in \mathbb{R}_+. \tag{10}$$

By the fact that g is exponentially increasing we also have

$$\sum_{i=1}^{E_h[x]} g(x - ih + h) \le \sum_{i=1}^{E_h[x]} \frac{g(x)}{\gamma^{i-1}} \le g(x)\frac{\gamma}{\gamma - 1}.$$

This, (8) and (10) imply that

$$f(x) - A(x) \in \left(2\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma - 1}\right) g(x)V,$$

which proves the assertion of (i).

(ii) Let $x \in \mathbb{R}_+$. We prove that

$$2^{n(x)+1} \le 4\frac{g(x)}{g(0)}. \tag{11}$$

At first suppose that $x \in \,]0, h[$. Then $n(x) = 0$, so $2^{n(x)+1} = 2 \le 4 \le 4\frac{g(x)}{g(0)}$, so (11) is trivial. Now let $x \in [h, \infty[$. As g is powerly increasing,

$$4g(x) \ge 4 \cdot 2^{n(x)-1} g\left(\frac{x}{2^{n(x)-1}}\right) \ge 2^{n(x)+1} g(0),$$

which yields (11). Moreover

$$\sum_{i=1}^{n(x)} 2^{i-1} g\left(\frac{x}{2^i} + \frac{x}{2^i}\right) \le \sum_{i=1}^{n} \frac{g(x)}{\gamma^{i-1}} \le \frac{\gamma}{\gamma - 1} g(x).$$

This, (9), and (11) impliy that

$$f(x) - A(x) \in \left(4\frac{g(h)}{g(0)} + \frac{\gamma}{\gamma - 1}\right) g(x)V,$$

which makes the proof of (ii) complete. □

Remark 1. One can check that every exponentially increasing function is powerly increasing. Therefore one may ask whether it makes sense in Theorem 2 to investigate both powerly increasing and exponentially increasing functions. The only reason to consider exponential functions is that it will allow us to obtain a better constant of approximation in the solution of *Problem A*.

Now we solve the *Problem A*.

Corollary 1. *Let $f :]0,1] \to E$ be a function such that*

$$f(xy) - xf(y) - yf(x) \in V \quad \text{for } x, y \in]0,1], \tag{12}$$

and let $z \in (0,1)$ be arbitrarily fixed. Then there exists a unique function $F_z :]0,1] \to E$ such that

$$F_z(z) = f(z), \tag{13}$$

$$F_z(xy) = xF_z(y) + yF_z(x), \tag{14}$$

and that

$$f(x) - F_z(x) \in K_z V \quad \text{for } x \in]0,1], \tag{15}$$

where $K_z := \frac{2}{z} + \frac{1}{1-z}$ (the minimal value of K_z is equal to $3 + 2\sqrt{2}$ and is obtained for $z = 2 - \sqrt{2}$).

Proof. For $K \subset \mathbb{R}$ we denote by E^K the vector space of all functions from $K \to E$. We define the linear operator $\mathcal{A} : E^{\,]0,1]} \to E^{\mathbb{R}_+}$ by the formula

$$\mathcal{A}(f)(x) := \exp(x) f(\exp(-x)) \quad \text{for } x \in \mathbb{R}_+.$$

The fact that f satisfies (12) is equivalent to

$$\mathcal{A}(f)(u+v) - \mathcal{A}(f)(u) - A(f)(v) \in \exp(u+v)V \quad \text{for } u, v \in \mathbb{R}_+.$$

Obviously exp is exponentially increasing with $h := -\exp^{-1}(z) = -\ln(z)$, $\gamma := \exp(h) = \frac{1}{z}$. Therefore by Theorem 2(i) there exists a unique $A_h : \mathbb{R}_+ \to E$ such that

$$A_h(h) = \mathcal{A}(f)(h), \tag{16}$$

$$A_h(u+v) = A_h(u) + A_h(v) \quad \text{for } u, v \in \mathbb{R}_+, \tag{17}$$

$$\mathcal{A}(f)(u) - A_h(u) \in K_z \exp(u) V, \tag{18}$$

where $K_z = 2\frac{\exp(h)}{\exp(0)} + \frac{\gamma}{\gamma - 1} = \frac{2}{z} + \frac{1}{1-z}$.

Let $F_z := \mathcal{A}^{-1}(A_h)$. Then one can easily check that (16), (17), (18) mean that F_z satisfies (12), (13), (14).

We show that F_z is unique. Suppose that there exists F_z' satisfying (16), (17), (18). Then $\mathcal{A}(F_z')$ satisfies (12), (13), (14), so $\mathcal{A}(F_z') = A_h = \mathcal{A}(F_z)$. As $\mathcal{A}$ is a bijection this implies that $F_z' = F_z$. $\square$

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