

A result on the neutrix convolution product of distributions

By BRIAN FISHER (Leicester)

The convolution product of two distributions satisfying certain conditions was defined by GELFAND and SHILOV [3] as follows.

Definition 1. Let f and g be distributions satisfying either of the following conditions:

- (a) either f or g has bounded support,
- (b) the supports of f and g are bounded on the same side.

Then the convolution product $f*g$ is defined by

$$((f*g)(x), \varphi(x)) = (g(y), (f(x), \varphi(x+y)))$$

for arbitrary test function φ in the space K of infinitely differentiable functions with compact support.

Note that if f has bounded support then $(f(x), \varphi(x+y))$ is in K and so $(g(y), (f(x), \varphi(x+y)))$ is meaningful. On the other hand, if g has bounded support or the supports of f and g are bounded on the same side, then the intersection of the supports of $g(y)$ and $(f(x), \varphi(x+y))$ is bounded and so $(g(y), (f(x), \varphi(x+y)))$ is again meaningful.

It follows that if the convolution product $f*g$ exists by this definition then

$$\begin{aligned} (1) \quad f*g &= g*f, \\ (f*g)' &= f*g' = f'*g. \end{aligned}$$

This definition of the convolution product is rather restrictive and the next definition was introduced in [2] in order to extend the convolution product to a larger class of distributions. In this definition, we denote the convolution product by $f*g$ to distinguish it from the convolution product given in definition 1.

Definition 2. Let f and g be distributions and let $f_n = f\tau_n$ for $n=1, 2, \dots$, where τ is an infinitely differentiable function satisfying the following conditions:

- (i) $\tau(x) = \tau(-x)$,
- (ii) $0 \leq \tau(x) \leq 1$,
- (iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
- (iv) $\tau(x) = 0$ for $|x| \geq 1$

and the function τ_n is defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

Then the convolution product $f \overline{[*]} g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, providing the limit h exists in the sense that

$$N\text{-}\lim_{n \rightarrow \infty} (f_n * g, \varphi) = (h, \varphi)$$

for all test functions φ in K , where N is the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, r = 1, 2, \dots)$$

and all functions $E(n)$ for which $\lim_{n \rightarrow \infty} E(n) = 0$.

Note that in this definition the convolution product $f_n * g$ is in the sense of definition 1, the distribution f_n having bounded support, since the support of τ_n is contained in the interval $(-n - n^{-n}, n + n^{-n})$. Note also that because the definition is nonsymmetric, the convolution product $f \overline{[*]} g$ is not always commutative.

The following theorem was proved in [2] and shows that definition 2 is a generalization of definition 1.

Theorem 1. *Let f and g be distributions satisfying either condition (a) or condition (b) of definition 1. Then the convolution product $f \overline{[*]} g$ exists and*

$$f \overline{[*]} g = f * g.$$

The next two theorems were also proved in [2].

Theorem 2. *Let f and g be distributions and suppose that the convolution product $f \overline{[*]} g$ exists. Then the convolution product $f \overline{[*]} g'$ exists and*

$$(f \overline{[*]} g)' = f \overline{[*]} g'.$$

Theorem 3. *The convolution product $x_+^\lambda \overline{[*]} x^s$ exists and*

$$x_+^\lambda \overline{[*]} x^s = 0$$

for $\lambda > -1$ and $s = 0, 1, 2, \dots$

In the following, $E(n)$ denotes any function such that $\lim_{n \rightarrow \infty} E(n) = 0$ and $0(n)$ denotes any function such that $N\text{-}\lim_{n \rightarrow \infty} 0(n) = 0$. We will therefore write

$$0(n) + 0(n) = 0(n) + E(n) = 0(n).$$

We now prove the following extension of Theorem 3.

Theorem 4. *The convolution product $x_+^\lambda \overline{[*]} x^s$ exists and*

$$x_+^\lambda \overline{[*]} x^s = 0$$

for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $s = 0, 1, 2, \dots$

PROOF. We will suppose that $-r-1 < \lambda < -r$. Then the distribution x_+^λ is defined by

$$x_+^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} (x_+^{\lambda+r})^{(r)},$$

where $\lambda+r > -1$.

The convolution product $(x_+^\lambda)_n * x^s$, where

$$(x_+^\lambda)_n = x_+^\lambda \tau_n(x),$$

exists by definition 1 and so

$$((x_+^\lambda)_n * x^s, \varphi(x)) = ((y_+^\lambda)_n, (x^s, \varphi(x+y)))$$

for arbitrary test function φ in K . If the support of φ is contained in the interval (a, b) , we have

$$\begin{aligned} (x^s, \varphi(x+y)) &= \int_{a-y}^{b-y} x^s \varphi(x+y) dx = \int_a^b (t-y)^s \varphi(t) dt = \\ &= \sum_{i=0}^s a_i y^i = p(y), \end{aligned}$$

where

$$a_i = \binom{s}{i} (-1)^i (t^{s-i}, \varphi(t))$$

for $i=0, 1, \dots, s$ and $\binom{s}{i}$ denotes the binomial coefficients.

Thus

$$\begin{aligned} ((x_+^\lambda)_n * x^s, \varphi(x)) &= ((y_+^\lambda)_n, p(y)) = (y_+^\lambda, (\tau_n p)(y)) = \\ &= \frac{(-1)^r \Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} (y_+^{\lambda+r}, (\tau_n p)^{(r)}(y)) = \\ &= \frac{(-1)^r \Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} \left[\int_0^n y^{\lambda+r} p^{(r)}(y) dy + \int_n^{n+n^{-n}} y^{\lambda+r} (\tau_n p)^{(r)}(y) dy \right]. \end{aligned}$$

It is easily seen that

$$\int_0^n y^{\lambda+r} p^{(r)}(y) dy = 0(n),$$

where $0(n)=0$ for $r>s$, and on noting that

$$\tau_n^{(i)}(n) = \begin{cases} 1, & i=0, \\ 0, & i \geq 1, \end{cases}$$

$$\tau_n^{(i)}(n+n^{-n}) = 0, \quad i \geq 0$$

we have

$$\begin{aligned} \int_n^{n+n^{-n}} y^{\lambda+r} (\tau_n p)^{(r)}(y) dy &= \int_n^{n+n^{-n}} y^{\lambda+r} d(\tau_n p)^{(r-1)}(y) = \\ &= [y^{\lambda+r} (\tau_n p)^{(r-1)}(y)]_n^{n+n^{-n}} - (\lambda+r) \int_n^{n+n^{-n}} y^{\lambda+r-1} (\tau_n p)^{(r-1)}(y) dy = \\ &= 0(n) + \frac{(-1)^r \Gamma(\lambda+r+1)}{\Gamma(\lambda+1)} \int_n^{n+n^{-n}} y^{\lambda} (\tau_n p)(y) dy. \end{aligned}$$

Now

$$\left| \int_n^{n+n^{-n}} y^{\lambda} (\tau_n p)(y) dy \right| \leq (s+1) \max \{|a_i| : 0 \leq i \leq s\} (n+n^{-n})^{s+\lambda} n^{-n} = E(n)$$

and so

$$\int_n^{n+n^{-n}} y^{\lambda+r} (\tau_n p)^{(r)}(y) dy = 0(n).$$

It follows that

$$((x_+^{\lambda})_n * x^s, \varphi(x)) = 0(n)$$

and so

$$N\text{-}\lim_{n \rightarrow \infty} ((x_+^{\lambda})_n * x^s, \varphi(x)) = 0 = (0, \varphi(x))$$

for $s=0, 1, 2, \dots$. The result of the theorem follows.

Corollary 1. *The convolution products $x_-^{\lambda} \overline{[*]} x^s$, $|x|^{\lambda} \overline{[*]} x^s$ and $(\operatorname{sgn} x \cdot |x|^{\lambda}) \overline{[*]} x^s$ exist and*

$$x_-^{\lambda} \overline{[*]} x^s = |x|^{\lambda} \overline{[*]} x^s = (\operatorname{sgn} x \cdot |x|^{\lambda}) \overline{[*]} x^s = 0$$

for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $s=0, 1, 2, \dots$

PROOF. The results follow from the theorem on noting that

$$x_-^{\lambda} = (-x)_+^{\lambda}, \quad |x|^{\lambda} = x_+^{\lambda} + x_-^{\lambda}, \quad \operatorname{sgn} x \cdot |x|^{\lambda} = x_+^{\lambda} - x_-^{\lambda}.$$

Corollary 2. *The convolution product $x_+^{\lambda} \overline{[*]} x_-^s$ exists and*

$$x_+^{\lambda} \overline{[*]} x_-^s = (-1)^{s+1} B(\lambda+1, s+1) x_+^{\lambda+s+1}$$

for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $s=0, 1, 2, \dots$, where B denotes the beta function.

PROOF. We will again suppose that $-r-1 < \lambda < -r$ and then

$$x_+^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} (x_+^{\lambda+r})^{(r)}.$$

The convolution product $x_+^{\lambda+r} * x_+^s$ exists by definition 1 and

$$\begin{aligned} x_+^{\lambda+r} * x_+^s &= \int_{-\infty}^{\infty} (x-t)_+^{\lambda+r} t_+^s dt = x_+^{\lambda+r+s+1} \int_0^1 (1-u)^{\lambda+r} u^s dt = \\ &= B(\lambda+r+1, s+1) x_+^{\lambda+r+s+1}. \end{aligned}$$

Equation (1) holds and it follows that

$$\frac{\Gamma(\lambda+r+1)}{\Gamma(\lambda+1)} x_+^{\lambda} * x_+^s = \frac{\Gamma(\lambda+r+1, s+2)}{\Gamma(\lambda+s+2)} B(\lambda+r+1, s+1) x_+^{\lambda+s+1}.$$

Thus

$$x_+^{\lambda} * x_+^s = B(\lambda+1, s+1) x_+^{\lambda+s+1}$$

for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $s=0, 1, 2, \dots$. The result of the corollary now follows on noting that

$$x_-^s = (-1)^s (x^s - x_+^s).$$

Corollary 3. The convolution products $x_-^{\lambda} \overline{[*]} x_+^s$, $|x|^{\lambda} \overline{[*]} x_+^s$ and $(\operatorname{sgn} x \cdot |x|^{\lambda}) \overline{[*]} x_+^s$ exists and

$$x_-^{\lambda} \overline{[*]} x_+^s = (-1)^{s+1} B(\lambda+1, s+1) x_+^{\lambda+s+1},$$

$$|x|^{\lambda} \overline{[*]} x_+^s = \begin{cases} B(\lambda+1, s+1) \operatorname{sgn} x \cdot |x|^{\lambda+s+1}, & \text{even } s, \\ B(\lambda+1, s+1) |x|^{\lambda+s+1}, & \text{odd } s, \end{cases}$$

$$(\operatorname{sgn} x \cdot |x|^{\lambda}) \overline{[*]} x_+^s = \begin{cases} B(\lambda+1, s+1) |x|^{\lambda+s+1}, & \text{even } s, \\ B(\lambda+1, s+1) \operatorname{sgn} x \cdot |x|^{\lambda+s+1}, & \text{odd } s \end{cases}$$

for $\lambda < -1$, $\lambda \neq -2, -3, \dots$ and $s=0, 1, 2, \dots$

The results of this corollary are immediate.

References

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
LEICESTER
LE1 7RH
ENGLAND

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