

π -holomorphically planar curves and π -holomorphically projective transformations

GEORGY MARKOV (Plovdiv, Bulgaria)

and

MILEVA PRVANOVIĆ (Novi Sad, Yugoslavia)

1. Introduction. Let A_{2n} be an affinely connected manifold of dimension $2n$ whose components of the affine connection are Γ_{jk}^i . Let A_{2n} be endowed with a tensor field $f_j^i \neq \delta_j^i$ satisfying

$$(1.1) \quad f_s^i f_j^s = \omega \delta_j^i, \quad \omega = +1 \quad \text{or} \quad -1.$$

If $\omega = -1$, the manifold is an almost complex space, and if $\omega = +1$, it is an almost product space. In the adapted coordinate systems, the matrix of f has the form:

$$(f_j^i) = \begin{pmatrix} \sqrt{\omega} f_b^a & 0 \\ 0 & -\sqrt{\omega} f_b^a \end{pmatrix},$$

where the indices a, b run over the range $1, 2, \dots, n$, while $i, j, k, \dots$, run over the range $1, 2, \dots, 2n$. Therefore

$$(1.2) \quad f_i^i = 0.$$

An affine connection Γ is called an f -connection if

$$(1.3) \quad \nabla_k f_j^i = 0$$

where ∇_k denotes the covariant derivative with respect to the connection Γ .

A curve $x^i = x^i(t)$ satisfying the differential equations

$$(1.4) \quad \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = \alpha(t) \frac{dx^i}{dt} + \beta(t) f_s^i \frac{dx^s}{dt}$$

is a holomorphically planar curve. We see directly from (1.4) that a curve is a holomorphically planar curve if and only if the plane elements determined by $\frac{dx^i}{dt}$

and $f_s^i \frac{dx^s}{dt}$ are parallel along the curve itself.

Two f -connections Γ and $\bar{\Gamma}$ are said to be H -projectively (holomorphically projectively) related to each other if they have all holomorphically planar curves in common.

H -planar curves and H -projective transformations have been studied by many authors ([1], [2], [3], [4], [5], [6]). So it is known that two symmetric f -connections Γ and $\bar{\Gamma}$ are H -projectively related if and only if

$$(1.5) \quad \bar{\Gamma}_{ks}^i = \Gamma_{ks}^i + \psi_k \delta_s^i + \psi_s \delta_k^i + \omega \psi_p f_k^p f_s^i + \omega \psi_p f_s^p f_k^i,$$

where ψ_i is an arbitrary covector field. Equations (1.5) give H -projective transformations of the symmetric f -connection.

The object of the present paper is to generalize the notion of holomorphically planar curves and holomorphically projective transformations using the notion of π -geodesic vector field given by RADZISZEWSKI [7].

Let be given a fixed symmetric differentiable tensor field π of type $(0, 2)$ in A_{2n} , satisfying the condition $\det(\pi_{ij}) \neq 0$. In this paper we suppose that π_{ij} is a pure tensor, i.e.

$$(1.6) \quad {}^*O_{ij}^{pq} \pi_{pq} = 0$$

where

$${}^*O_{ij}^{pq} = \frac{1}{2} (\delta_i^p \delta_j^q - \omega f_i^p f_j^q).$$

The condition (1.6) is equivalent with

$$(1.7) \quad f_k^p \pi_{pj} = f_j^p \pi_{pk}.$$

In fact, (1.6) means

$$\pi_{ij} = \omega f_i^p f_j^q \pi_{pq},$$

from which, transvecting by f_k^i and taking into account the symmetry of π_{ij} , we get (1.7).

It is easy to prove that the tensor field $\tilde{\pi}^{ij}$, determined by

$$\pi_{sj} \tilde{\pi}^{si} = \delta_j^i$$

is also a pure tensor field, that is

$$(1.8) \quad f_s^i \tilde{\pi}^{sj} = f_s^j \tilde{\pi}^{is}.$$

According to the definition given by RADZISZEWSKI [7], a vector field w is called π -geodesic if

$$(1.9) \quad \nabla_k (\pi_{is} w^s) w^k = \lambda \pi_{is} w^s.$$

Integral curves of a π -geodesic vector field are called π -geodesics. Their differential equations are [7], [8]:

$$\frac{d^2 x^i}{dt^2} + [\Gamma_{ks}^i + (\nabla_k \pi_{ps}) \tilde{\pi}^{pi}] \frac{dx^k}{dt} \frac{dx^s}{dt} = \lambda \frac{dx^i}{dt};$$

In this paper we consider a vector field w satisfying, instead of (1.9), the condition

$$(1.10) \quad \nabla_k(\pi_{ip}w^p)w^k = \pi_{ip}(\alpha(t)w^p + \beta(t)f_q^p w^q).$$

We shall define, in § 2, πH -planar curves and πH -projective transformations and we shall prove in §§ 3 and 4 the existence of two different πH -projective curvature tensors, i.e. the existence of two different tensors each of which is invariant with respect to πH -projective transformations. One of them, called the second πH -projective curvature tensor, can be obtained directly from the H -projective curvature tensor.

2. πH -projective transformation. We have from (1.10)

$$(\nabla_k \pi_{ip})w^p w^k + \pi_{ip} \left(\frac{\partial w^p}{\partial x^k} + \Gamma_{ks}^p w^s \right) w^k = \pi_{ip}(\alpha w^p + \beta f_q^p w^q),$$

so that, transvecting with $\tilde{\pi}^{ih}$, we get

$$(2.1) \quad \frac{\partial w^h}{\partial x^k} + [\Gamma_{ks}^h + (\nabla_k \pi_{ip})\tilde{\pi}^{ph}]w^s w^k = \alpha w^h + \beta f_q^h w^q.$$

Definition. We say that the integral curves of the vector field w satisfying (2.1) are π -holomorphically planar curves, abbreviated to πH -planar curves.

If $x^i = x^i(t)$ are πH -planar curves, then $w^h = \frac{dx^h}{dt}$, and we obtain from (2.1) that

$$(2.2) \quad \frac{d^2 x^h}{dt^2} + [\Gamma_{ks}^h + (\nabla_k \pi_{ps})\tilde{\pi}^{pk}] \frac{dx^s}{dt} \frac{dx^k}{dt} = \alpha \frac{dx^h}{dt} + \beta f_p^h \frac{dx^p}{dt}$$

are their differential equations.

Definition. A πH -projective transformation is a change of connection Γ which does not change the system of πH -planar curves.

To find a πH -projective transformation, we first notice that (1.4) and (2.2) show that a πH -planar curve with respect to the connection Γ is a holomorphically planar curve with respect to the connection

$$(2.3) \quad G_{ks}^i = \Gamma_{ks}^i + (\nabla_k \pi_{ps})\tilde{\pi}^{pi}.$$

If we suppose that G_{ks}^i is a symmetric f -connection, a change of G_{ks}^i which does not change the system of holomorphically planar curves has the form (1.5). Thus we have

$$(2.4) \quad \bar{G}_{ks}^i = G_{ks}^i + \psi_k \delta_s^i + \psi_s \delta_k^i + \omega(\psi_p f_k^p f_s^i + \psi_p f_s^p f_k^i).$$

As for $\bar{G}_{ks}^i$, it must be a symmetric f -connection too, and must also determine πH -planar curves, i.e. it must have the form

$$(2.5) \quad \bar{G}_{ks}^i = \bar{\Gamma}_{ks}^i + (\bar{\nabla}_k \pi_{ps})\tilde{\pi}^{pi},$$

where $\bar{\nabla}$ denotes the covariant derivative with respect to the connection $\bar{\Gamma}$.

From (2.3) and (2.5), we have

$$G_{ks}^i = \frac{\partial \pi_{as}}{\partial x^k} \tilde{\pi}^{ai} - \Gamma_{ka}^p \pi_{ps} \tilde{\pi}^{ai},$$

$$\bar{G}_{ks}^i = \frac{\partial \pi_{as}}{\partial x^k} \tilde{\pi}^{ai} - \bar{\Gamma}_{ka}^p \pi_{ps} \tilde{\pi}^{ai}.$$

Consequently

$$\bar{G}_{ks}^i - G_{ks}^i = -(\bar{\Gamma}_{ka}^p - \Gamma_{ka}^p) \pi_{ps} \tilde{\pi}^{ai},$$

from which, transvecting with $\tilde{\pi}^{st} \pi_{ij}$, we obtain

$$\bar{\Gamma}_{kj}^t = \Gamma_{kj}^t - (\bar{G}_{ks}^i - G_{ks}^i) \tilde{\pi}^{st} \pi_{ij}.$$

Substituting $\bar{G}_{ks}^i - G_{ks}^i$ from (2.4), we get

$$(2.6) \quad \bar{\Gamma}_{kj}^t = \Gamma_{kj}^t - \psi_s \tilde{\pi}^{st} \pi_{kj} - \delta_j^t \psi_k - \omega \psi_a (f_s^a f_k^i + f_k^a f_s^i) \tilde{\pi}^{st} \pi_{ij}.$$

Conversely, let us suppose that (2.6) holds good. Then we have

$$\begin{aligned} \bar{G}_{kj}^i &= \frac{\partial \pi_{aj}}{\partial x^k} \tilde{\pi}^{ai} - \bar{\Gamma}_{ka}^p \pi_{pj} \tilde{\pi}^{ai} = \\ &= \frac{\partial \pi_{aj}}{\partial x^k} \tilde{\pi}^{ai} - \Gamma_{ka}^p \pi_{pj} \tilde{\pi}^{ai} + \psi_j \delta_k^i + \psi_k \delta_j^i + \omega \psi_b (f_k^b f_j^i + f_j^b f_k^i) = \\ &= G_{kj}^i + \psi_j \delta_k^i + \psi_k \delta_j^i + \omega \psi_b (f_k^b f_j^i + f_j^b f_k^i), \end{aligned}$$

that is, we can express the connection (2.5) in the form (2.4). Thus, we have the following

Theorem. *The condition (2.6) is necessary and sufficient for the connections $\bar{\Gamma}$ and Γ to have all πH -planar curves in common.*

Taking into account (1.7), the relation (2.6) can be rewritten in the following form

$$(2.7) \quad \bar{\Gamma}_{kj}^t = \Gamma_{kj}^t - \psi_s \tilde{\pi}^{st} \pi_{kj} - \delta_j^t \psi_k - \omega \psi_a f_k^a f_j^t - \omega \psi_a f_s^a f_k^i \tilde{\pi}^{st} \pi_{ij}.$$

Definition. The correspondence (2.7) is called a πH -projective transformation.

It was already mentioned that (1.5) takes place for two symmetric f -connections. This means that both connections $\bar{G}$ and G in (2.4) are symmetric f -connections. But the symmetry of the connection G implies

$$(2.8) \quad \Gamma_{ks}^i - \Gamma_{sk}^i + (\nabla_k \pi_{as} - \nabla_s \pi_{ak}) \tilde{\pi}^{ai} = 0,$$

because of (2.3). In the case when Γ is a symmetric connection too, this condition reduces to

$$(2.9) \quad \nabla_k \pi_{as} - \nabla_s \pi_{ak} = 0,$$

and the connection $\bar{\Gamma}$ cannot be symmetric. Therefore, it must satisfy the condition

$$\bar{\Gamma}_{ks}^i - \bar{\Gamma}_{sk}^i + (\bar{\nabla}_k \pi_{as} - \bar{\nabla}_s \pi_{ak}) \tilde{\pi}^{ai} = 0.$$

On the other hand, both connections $\bar{\Gamma}$ and Γ are f -connections. This statement follows from the following

Theorem. *If G is a f -connection, the connection Γ is an f -connection, too.*

PROOF. If G is an f -connection, we have

$$\frac{\partial f_r^i}{\partial x^k} + G_{ks}^i f_r^s - G_{kr}^i f_s^i = 0,$$

or, substituting (2.3) and taking into account (1.8),

$$(2.10) \quad \nabla_k f_r^i + (\nabla_k \pi_{as}) \tilde{\pi}^{ai} f_r^s - (\nabla_k \pi_{ra}) \tilde{\pi}^{is} f_s^a = 0.$$

On the other hand

$$\nabla_k (\pi_{sa} f_r^s) \tilde{\pi}^{ai} = [(\nabla_k \pi_{sa}) f_r^s + (\nabla_k f_r^s) \pi_{sa}] \tilde{\pi}^{ai},$$

from which, using (1.7), we get

$$(\nabla_k \pi_{sa}) \tilde{\pi}^{ai} f_r^s = \nabla_k (\pi_{sr} f_a^s) \tilde{\pi}^{ai} - \nabla_k f_r^i = (\nabla_k \pi_{sr}) f_a^s \tilde{\pi}^{ai} + (\nabla_k f_a^s) \pi_{sr} \tilde{\pi}^{ai} - \nabla_k f_r^i.$$

Substituting this into (2.10), we obtain

$$(\nabla_k f_a^s) \pi_{sr} \tilde{\pi}^{ai} = 0,$$

from which, transvecting with $\pi_{ij} \tilde{\pi}^{rt}$, we get

$$(2.11) \quad \nabla_k f_j^i = 0.$$

This proves the theorem.

3. The first πH -projective curvature tensor. Now, we shall consider the curvature tensor

$$\bar{R}^i_{rkj} = \frac{\partial \bar{\Gamma}^i_{jr}}{\partial x^k} - \frac{\partial \bar{\Gamma}^i_{kr}}{\partial x^j} + \bar{\Gamma}^i_{kq} \bar{\Gamma}^q_{jr} - \bar{\Gamma}^i_{jq} \bar{\Gamma}^q_{kr}.$$

After some computation, using (2.7), we get

$$\begin{aligned} \bar{R}^i_{rkj} = & R^i_{rkj} - \delta_r^i \left(\frac{\partial \psi_j}{\partial x^k} - \frac{\partial \psi_k}{\partial x^j} \right) - \omega f_r^i \left[\frac{\partial}{\partial x^k} (\psi_a f_j^a) - \frac{\partial}{\partial x^j} (\psi_a f_k^a) \right] - \\ & - \pi_{jr} [\nabla_k (\psi_a \tilde{\pi}^{ai}) - \psi_a \tilde{\pi}^{ai} \psi_k - \omega \psi_a f_k^a f_q^i \tilde{\pi}^{bq} \psi_b] + \\ & + \pi_{kr} [\nabla_j (\psi_a \tilde{\pi}^{ai}) - \psi_a \tilde{\pi}^{ai} \psi_j - \omega \psi_a f_j^a f_q^i \tilde{\pi}^{bq} \psi_b] - \\ & - \omega f_j^i \pi_{tr} [\nabla_k (\psi_a f_p^a \tilde{\pi}^{pi}) - \psi_b f_s^b \tilde{\pi}^{si} \psi_k - \omega \psi_a f_k^a f_q^i \tilde{\pi}^{sq} \psi_b f_s^b] + \\ & + \omega f_k^i \pi_{tr} [\nabla_j (\psi_a f_p^a \tilde{\pi}^{pi}) - \psi_b f_s^b \tilde{\pi}^{si} \psi_j - \omega \psi_a f_j^a f_q^i \tilde{\pi}^{sq} \psi_b f_s^b] + \\ & + \psi_a \tilde{\pi}^{ai} [\nabla_j \pi_{kr} - \nabla_k \pi_{jr} + (\Gamma_{jk}^q - \Gamma_{kj}^q) \pi_{qr}] + \\ & + \omega \psi_a f_p^a \tilde{\pi}^{pi} [\nabla_j (f_k^t \pi_{tr}) - \nabla_k (f_j^t \pi_{tr}) + (\Gamma_{jk}^q - \Gamma_{kj}^q) f_q^t \pi_{tr}], \end{aligned}$$

where R^i_{rkj} are the components of the curvature tensor of the connection Γ .

Transvecting (2.8) with π_{ij} , we find

$$(\Gamma_{ks}^i - \Gamma_{sk}^i) \pi_{ij} + \nabla_k \pi_{js} - \nabla_s \pi_{jk} = 0.$$

On the other hand, transvecting (2.8) with $f_i^b \pi_{br}$ and taking into account (1.7) and (1.8), we have

$$(\Gamma_{ks}^i - \Gamma_{sk}^i) f_i^b \pi_{br} + \nabla_k (f_r^a \pi_{as}) - \nabla_s (f_r^a \pi_{ak}) = 0.$$

Therefore

$$(3.1) \quad \begin{aligned} \bar{R}_{rkj}^i = R_{rkj}^i - \delta_r^i \left(\frac{\partial \psi_j}{\partial x^k} - \frac{\partial \psi_k}{\partial x^j} \right) - \omega f_r^i \left[\frac{\partial}{\partial x_k} (\psi_a f_j^a) - \frac{\partial}{\partial x^j} (\psi_a f_k^a) \right] - \\ - \pi_{jr} \nabla_k (\psi_a \tilde{\pi}^{ai}) - \psi_a \tilde{\pi}^{ai} \psi_k - \omega \psi_a f_k^a f_q^i \tilde{\pi}^{bq} \psi_b + \\ + \pi_{kr} [\nabla_j (\psi_a \tilde{\pi}^{ai}) - \psi_a \tilde{\pi}^{ai} \psi_j - \omega \psi_a f_j^a f_q^i \tilde{\pi}^{bq} \psi_b] - \\ - \omega f_j^i \pi_{tr} [\nabla_k (\psi_a f_p^a \tilde{\pi}^{pi}) - \psi_b f_s^b \tilde{\pi}^{si} \psi_k - \omega \psi_a f_k^a f_q^i \tilde{\pi}^{sq} \psi_b f_s^b] + \\ + \omega f_k^i \pi_{tr} [\nabla_j (\psi_a f_p^a \tilde{\pi}^{pi}) - \psi_b f_s^b \tilde{\pi}^{si} \psi_j - \omega \psi_a f_j^a f_q^i \tilde{\pi}^{sq} \psi_b f_s^b]. \end{aligned}$$

Let us put

$$(3.2) \quad \theta_k^i = \nabla_k (\psi_a \tilde{\pi}^{ai}) - \psi_a \tilde{\pi}^{ai} \psi_k - \omega \psi_a f_k^a f_q^i \tilde{\pi}^{bq} \psi_b,$$

$$(3.3) \quad \varphi_{jk} = \frac{\partial \psi_k}{\partial x^j} - \frac{\partial \psi_j}{\partial x^k},$$

$$(3.4) \quad \eta_{jk} = \frac{\partial}{\partial x^j} (\psi_a f_k^a) - \frac{\partial}{\partial x^k} (\psi_a f_j^a).$$

Then, taking into account (1.8) and (2.11), we have

$$\nabla_k (\psi_a f_p^a \tilde{\pi}^{pi}) - \psi_b f_s^b \tilde{\pi}^{si} \psi_k - \omega \psi_a f_k^a f_q^i \tilde{\pi}^{sq} \psi_b f_s^b = f_a^i \theta_k^a.$$

This means that (3.4) may be written in the form

$$(3.5) \quad \begin{aligned} \bar{R}_{rkj}^i = R_{rkj}^i + \delta_r^i \varphi_{jk} + \omega f_r^i \eta_{jk} - \\ - \pi_{jr} \theta_k^i + \pi_{kr} \theta_j^i - \omega f_j^a \pi_{ar} f_p^i \theta_k^p + \omega f_k^a \pi_{ar} f_p^i \theta_j^p. \end{aligned}$$

It is easily seen that

$$-\pi_{ja} \theta_k^a = -\nabla_k \psi_j - \psi_b \pi_{ja} \nabla_k \tilde{\pi}^{ba} - \psi_j \psi_k - \omega \psi_a \psi_b f_k^a f_j^b.$$

From $\pi_{ja} \tilde{\pi}^{ba} = \delta_j^b$, we have

$$-\pi_{ja} \nabla_k \tilde{\pi}^{ba} = (\nabla_k \pi_{ja}) \tilde{\pi}^{ba}.$$

Therefore

$$-\pi_{ja} \theta_k^a = -\frac{\partial \psi_j}{\partial x^k} + \Gamma_{kj}^s \psi_s + \psi_b \tilde{\pi}^{ba} (\nabla_k \pi_{ja}) - \psi_j \psi_k - \omega \psi_a \psi_b f_k^a f_j^b$$

and

$$-\pi_{ja} \theta_k^a + \pi_{ka} \theta_j^a = \frac{\partial \psi_k}{\partial x^j} - \frac{\partial \psi_j}{\partial x^k} + [\Gamma_{kj}^s - \Gamma_{jk}^s + \tilde{\pi}^{sa} (\nabla_k \pi_{ja} - \nabla_j \pi_{ka})] \psi_s,$$

from which

$$(3.6) \quad \pi_{ka}\theta_j^a - \pi_{ja}\theta_k^a = \varphi_{jk},$$

because of (2.8) and (3.3).

Similarly

$$\begin{aligned} -\pi_{js}f_a^s\theta_k^a &= -\frac{\partial}{\partial x^k}(\psi_b f_j^b) + \psi_b \frac{\partial f_j^b}{\partial x^k} + f_j^b \Gamma_{kb}^s \psi_s + \\ &+ \psi_b f_j^s (\nabla_k \pi_{as}) \tilde{\pi}^{ba} + \psi_b f_j^b \psi_k + \psi_b f_k^b \psi_j. \end{aligned}$$

From (2.11) we obtain

$$\psi_b \frac{\partial f_j^b}{\partial x^k} = -\psi_s \Gamma_{kb}^s f_j^b + \psi_b f_s^b \Gamma_{kj}^s.$$

Substituting this into the preceding relation, we get

$$-\pi_{js}f_a^s\theta_k^a = -\frac{\partial}{\partial x^k}(\psi_b f_j^b) + \psi_b f_s^b \Gamma_{kj}^s + \psi_b f_j^s (\nabla_k \pi_{as}) \tilde{\pi}^{ba} + \psi_b f_j^b \psi_k + \psi_b f_k^b \psi_j.$$

Thus

$$(3.7) \quad \pi_{ks}f_a^s\theta_j^a - \pi_{js}f_a^s\theta_k^a = \eta_{jk}$$

because of (2.8) and (3.4).

Now we shall determine the functions θ , φ and η .

Contracting (3.5) with respect to i and r and taking into account (1.2), (1.7) and (3.6), we find

$$(3.8) \quad \varphi_{jk} = \frac{1}{2(n+1)} (\bar{R}_{akj}^a - R_{akj}^a).$$

Also, we have from (3.5)

$$\bar{R}_{rkj}^a f_a^i = R_{rkj}^a f_a^i + f_r^i \varphi_{jk} + \delta_r^i \eta_{jk} + \pi_{kr} f_a^i \theta_j^a - \pi_{jr} f_a^i \theta_k^a + f_k^a \pi_{ar} \theta_j^i - f_j^a \pi_{ar} \theta_k^i,$$

from which, contracting with respect to i and r and taking into account (1.2), (1.7) and (3.4), we get

$$(3.9) \quad \eta_{jk} = \frac{1}{2(n+1)} (\bar{R}_{skj}^a - R_{skj}^a) f_a^s.$$

Contracting (3.5) with respect to i and j , we obtain

$$(3.10) \quad \bar{R}_{rk} = R_{rk} + \varphi_{rk} + \omega f_r^a \eta_{ak} - 2\pi_{ar} \theta_k^a + \pi_{kr} \theta_a^a + \omega f_k^a \pi_{ar} f_p^s \theta_s^p,$$

from which, transvecting with $\tilde{\pi}^{rk}$, we find

$$\bar{R}_{rk} \tilde{\pi}^{rk} = R_{rk} \tilde{\pi}^{rk} + \varphi_{rk} \tilde{\pi}^{rk} + \omega f_r^a \tilde{\pi}^{rk} \eta_{ak} + 2(n-1) \theta_a^a.$$

Here, R_{rk} and $\bar{R}_{rk}$ are Ricci tensors of connections Γ and $\bar{\Gamma}$ respectively.

Since φ_{rk} is a skew-symmetric tensor, while $\tilde{\pi}^{rk}$ is symmetric, $\varphi_{rk} \tilde{\pi}^{rk} = 0$. Similarly $f_r^a \tilde{\pi}^{rk} \eta_{ak} = 0$. Therefore, the preceding relation reduces to

$$(3.11) \quad \theta_a^a = \frac{1}{2(n-1)} (\bar{R}_{rk} - R_{rk}) \tilde{\pi}^{rk}.$$

Also, we find from (3.10)

$$\bar{R}_{rb} f_k^b = R_{rb} f_k^b + \varphi_{rb} f_k^b + \omega f_r^a f_k^b \eta_{ab} + \pi_{br} f_k^b \theta_a^a - 2\pi_{ar} f_k^b \theta_b^a + \pi_{kr} f_b^a \theta_a^b.$$

Transvecting with $\tilde{\pi}^{rk}$, we get

$$(3.12) \quad f_a^b \theta_b^a = \frac{1}{2(n-1)} (\bar{R}_{rb} - R_{rb}) f_k^b \tilde{\pi}^{rk}.$$

Transvecting (3.10) with $\tilde{\pi}^{ir}$, we find

$$\theta_k^i = \frac{1}{2} (-\bar{R}_{rk} \tilde{\pi}^{ri} + R_{rk} \tilde{\pi}^{ri} + \varphi_{ak} \tilde{\pi}^{ai} + \omega f_b^a \tilde{\pi}^{bi} \eta_{ak} + \delta_k^i \theta_a^a + \omega f_k^i f_b^a \theta_b^a)$$

so that, substituting from (3.8), (3.9), (3.11) and (3.12), we get

$$(3.13) \quad \begin{aligned} \theta_k^i = & \frac{1}{2} (-\bar{R}_{ar} + R_{ar}) \tilde{\pi}^{ai} + \frac{1}{4(n+1)} (\bar{R}_{ska}^s - R_{ska}^s) \tilde{\pi}^{ai} + \\ & + \frac{\omega}{4(n+1)} (\bar{R}_{ska}^t - R_{ska}^t) f_t^s f_b^a \tilde{\pi}^{bi} + \\ & + \frac{1}{4(n-1)} [\delta_k^i (\bar{R}_{ab} - R_{ab}) \tilde{\pi}^{ab} + \omega f_k^i (\bar{R}_{as} - R_{as}) f_b^s \tilde{\pi}^{ab}]. \end{aligned}$$

If we put

$$(3.14) \quad O_{ij}^{ab} = \frac{1}{2} (\delta_i^a \delta_j^b + \omega f_i^a f_j^b),$$

we can express (3.5) in the form

$$\bar{R}_{rkj}^i = R_{rkj}^i + \delta_r^i \varphi_{jk} + \omega f_r^i \eta_{jk} + 2O_{kp}^{ai} \pi_{ar} \theta_j^p - 2O_{jp}^{ai} \tilde{\pi}_{ar} \theta_k^p.$$

Thus, substituting (3.8), (3.9) and (3.13), we find

$$\begin{aligned} & \bar{R}_{rkj}^i - \frac{1}{2(n+1)} (\delta_r^i \bar{R}_{skj}^s + \omega f_r^i \bar{R}_{skj}^b f_b^s) + \\ & + O_{kp}^{qi} \pi_{qr} \left[\bar{R}_{aj} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} \bar{R}_{sjb}^t O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} \bar{R}_{as} O_{jb}^{ps} \right] - \\ & - O_{jp}^{qi} \pi_{qr} \left[\bar{R}_{ak} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} \bar{R}_{skb}^t O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} \bar{R}_{as} O_{kb}^{ps} \right] = \\ & = R_{rkj}^i - \frac{1}{2(n+1)} (\delta_r^i R_{skj}^s + \omega f_r^i R_{skj}^b f_b^s) + \\ & + O_{kp}^{qi} \pi_{qr} \left[R_{aj} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} R_{sjb}^t O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} R_{as} O_{jb}^{ps} \right] - \\ & - O_{jp}^{qi} \pi_{qr} \left[R_{ak} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} R_{skb}^t O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} R_{as} O_{kb}^{ps} \right]. \end{aligned}$$

Thus, we have the following

Theorem. *The tensor*

$$(3.15) \quad \begin{aligned} P_{rkj}^i &= R_{rkj}^i - \frac{1}{2(n+1)} (\delta_r^i R_{skj}^s + \omega f_r^i R_{skj}^b f_b^s) + \\ &+ O_{kp}^{qi} \pi_{qr} \left[R_{aj} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} R_{sjb}^t O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} R_{as} O_{jb}^{ps} \right] - \\ &- O_{jp}^{qi} \pi_{qr} \left[R_{ak} \tilde{\pi}^{ap} - \frac{1}{n+1} \tilde{\pi}^{ap} R_{skb}^t O_{ta}^{sb} - \frac{1}{n-1} \tilde{\pi}^{ab} R_{as} O_{kb}^{ps} \right] \end{aligned}$$

is invariant with respect to the πH -projective transformation (2.7).

We shall call this tensor the first πH -projective curvature tensor.

4. The second πH -projective curvature tensor. Taking into account (3.6) and (3.7), we can express (3.5) in the form

$$(4.1) \quad \begin{aligned} \bar{R}_{rkj}^i &= R_{rkj}^i + \delta_r^i (\pi_{ka} \theta_j^a - \pi_{ja} \theta_k^a) + \omega f_r^i (\pi_{ks} f_a^s \theta_j^a - \pi_{js} f_a^s \theta_k^a) + \\ &+ \pi_{kr} \theta_j^i - \pi_{jr} \theta_k^i + \omega f_k^a \pi_{ar} f_p^i \theta_j^p - \omega f_j^a \pi_{ar} f_p^i \theta_k^p. \end{aligned}$$

Therefore

$$(4.2) \quad {}^* \bar{R}_k^i = {}^* R_k^i - (2n+1) \theta_k^i + \tilde{\pi}^{ib} \pi_{ka} \theta_b^a + \omega \tilde{\pi}^{ab} f_a^i \pi_{ks} f_p^s \theta_b^p + \omega f_k^b f_a^i \theta_b^a,$$

where we have put

$$(4.3) \quad {}^* \bar{R}_k^i = \tilde{\pi}^{ab} \bar{R}_{akb}^i, \quad {}^* R_k^i = \tilde{\pi}^{ab} R_{akb}^i.$$

From (4.2), we get

$${}^* \bar{R}_k^a \pi_{ai} = {}^* R_k^a \pi_{ai} - (2n+1) \theta_k^a \pi_{ai} + \pi_{ka} \theta_i^a + \omega f_i^a f_k^b (\pi_{bs} \theta_a^s + \pi_{as} \theta_b^s),$$

which may be written as

$$(4.4) \quad {}^* \bar{R}_k^a \pi_{ai} = {}^* R_k^a \pi_{ai} - 2(n+1) \theta_k^a \pi_{ai} + 2O_{ik}^{ab} (\pi_{as} \theta_b^s + \pi_{bs} \theta_a^s).$$

If we put in (4.4) q instead of k , p instead of i , transvect with O_{ik}^{pq} and take into account that $O.O=O$, we get

$$({}^* \bar{R}_q^a \pi_{ap} - {}^* R_q^a \pi_{ap}) O_{ik}^{pq} = -2(n+1) \theta_q^a \pi_{ap} O_{ik}^{pq} + 2O_{ik}^{ab} (\pi_{as} \theta_b^s + \pi_{bs} \theta_a^s).$$

Changing indices and adding, we obtain

$$\begin{aligned} &-2(\theta_q^a \pi_{ap} + \theta_p^a \pi_{aq}) O_{ik}^{pq} = \\ &= \frac{1}{n-1} [({}^* \bar{R}_q^a \pi_{ap} + {}^* \bar{R}_p^a \pi_{aq}) - ({}^* R_q^a \pi_{ap} + {}^* R_p^a \pi_{aq})] O_{ik}^{pq}. \end{aligned}$$

Substituting this into (4.4), we get

$$(4.5) \quad \theta_k^a \pi_{ai} = -\bar{S}_{ki} + S_{ki} \quad \text{and} \quad \theta_k^i = (-\bar{S}_{ka} + S_{ka}) \tilde{\pi}^{ai},$$

where

$$(4.6) \quad \begin{aligned} \bar{S}_{ki} &= \frac{1}{2(n+1)} \left[{}^* \bar{R}^a_k \pi_{ai} + \frac{1}{n-1} ({}^* \bar{R}^a_q \pi_{aq} + {}^* \bar{R}^a_p \pi_{ap}) O_{ik}^{pq} \right], \\ S_{ki} &= \frac{1}{2(n+1)} \left[{}^* R^a_k \pi_{ai} + \frac{1}{n-1} ({}^* R^a_q \pi_{aq} + {}^* R^a_p \pi_{ap}) O_{ik}^{pq} \right]. \end{aligned}$$

Substituting (4.5) into (4.1), we find

$$(4.7) \quad \begin{aligned} & \bar{R}^i_{rkj} - \delta^i_r (\bar{S}_{kj} - \bar{S}_{jk}) + \omega f^i_r (f^a_k \bar{S}_{ja} - f^a_j \bar{S}_{ka}) + \\ & + (\pi_{kr} \bar{S}_{ja} - \pi_{jr} \bar{S}_{ka}) \tilde{\pi}^{ai} + \omega f^i_p \tilde{\pi}^{pb} \pi_{ar} (f^a_k \bar{S}_{jb} - f^a_j \bar{S}_{kb}) = \\ & = R^i_{rkj} - \delta^i_r (S_{kj} - S_{jk}) + \omega f^i_r (f^a_k S_{ja} - f^a_j S_{ka}) + \\ & + (\pi_{kr} S_{ja} - \pi_{jr} S_{ka}) \tilde{\pi}^{ai} + \omega f^i_p \tilde{\pi}^{pb} \pi_{ar} (f^a_k S_{jb} - f^a_j S_{kb}). \end{aligned}$$

Thus we get this

Theorem. *The tensor*

$$(4.8) \quad \begin{aligned} P^i_{rkj} &= R^i_{rkj} - \delta^i_r (S_{kj} - S_{jk}) + \omega f^i_r (f^a_k S_{ja} - f^a_j S_{ka}) + \\ & + (\pi_{kr} S_{ja} - \pi_{jr} S_{ka}) \tilde{\pi}^{ai} + \omega f^i_p \tilde{\pi}^{pb} \pi_{ar} (f^a_k S_{jb} - f^a_j S_{kb}). \end{aligned}$$

is invariant with respect to the πH -projective transformations (2.7).

We obtained the tensors (3.15) and (4.8) starting from the same relation (3.5). Thus, it is natural to ask: are they two different tensors? To answer the question, we shall discuss the case of a special space, namely the space satisfying $R^a_{akj} = 0$. Then

$$P^a_{akj} = {}^* R^a_j \pi_{ak} - {}^* R^a_k \pi_{aj} = (R^a_{pjq} \pi_{ak} - R^a_{pkq} \pi_{aj}) \tilde{\pi}^{pq}.$$

On the other hand, we find from (3.15) that

$$P^a_{1akj} = R_{kj} - R_{jk} + \frac{1}{2(n+1)} \omega (R^t_{skb} f^b_j - R^t_{sjb} f^b_k) f^s_t.$$

Thus

$$P^a_{2akj} \neq P^a_{1akj}.$$

But if

$$P^i_{2rkj} = P^i_{1rkj}$$

it would be

$$P^a_{2akj} = P^a_{1akj}.$$

This proves that P_1 and P_2 are two different tensors.

We shall call the tensor (4.8) the second πH -projective curvature tensor.

The relation (4.7) can also be obtained in the following way.

Let G^i_{rkj} indicate the components of the curvature tensor of the connection G . By straightforward computation, using (2.3), we find

$$(4.9) \quad G^i_{rkj} = -R^b_{akj} \pi_{br} \tilde{\pi}^{ai},$$

from which

$$(4.10) \quad G_{rk} = G^a_{rka} = -{}^*R^b_k \pi_{br}.$$

Similarly

$$(4.11) \quad \bar{G}^i_{rkj} = -\bar{R}^b_{akj} \pi_{br} \tilde{\pi}^{ai},$$

and

$$(4.12) \quad \bar{G}_{rk} = -{}^*\bar{R}^b_k \pi_{br},$$

where $\bar{G}^i_{rkj}$ are the components of the curvature tensor of the connection $\bar{G}$.

On the other hand, we know ([2], [3], [4] p. 263, [5]) that two symmetric f -connections, $\bar{G}$ and G , which are related to each other as in (2.4), have the H -projective curvature tensor in common, that is

$$\begin{aligned} & \bar{G}^i_{rkj} + \delta^i_r (\bar{P}_{kj} - \bar{P}_{jk}) + \omega f^i_r (f^a_j \bar{P}_{ka} - f^a_k \bar{P}_{ja}) + \\ & + \delta^i_j \bar{P}_{kr} - \delta^i_k \bar{P}_{jr} + \omega f^i_j f^a_r \bar{P}_{ka} - \omega f^i_k f^a_r \bar{P}_{ja} = \\ & = G^i_{rkj} + \delta^i_r (P_{kj} - P_{jk}) + \omega f^i_r (f^a_j P_{ka} - f^a_k P_{ja}) + \\ & + \delta^i_j P_{kr} - \delta^i_k P_{jr} + \omega f^i_j f^a_r P_{ka} - \omega f^i_k f^a_r P_{ja}, \end{aligned}$$

where

$$\begin{aligned} P_{kr} &= -\frac{1}{2(n+1)} \left[G_{rk} + \frac{1}{n-1} (G_{ab} + G_{ba}) O^{ab}_{rk} \right], \\ \bar{P}_{kr} &= -\frac{1}{2(n+1)} \left[\bar{G}_{rk} + \frac{1}{n-1} (\bar{G}_{ab} + \bar{G}_{ba}) O^{ab}_{rk} \right]. \end{aligned}$$

(Let us remember that the dimension of our space is $2n$.)

Substituting from (4.9) and (4.11), we find.

$$\begin{aligned} & R^b_{akj} \pi_{br} \tilde{\pi}^{ai} - \delta^i_r (P_{kj} - P_{jk}) - \omega f^i_r (f^a_j P_{ka} - f^a_k P_{ja}) - \\ & - \delta^i_j P_{kr} + \delta^i_k P_{jr} - \omega f^i_j f^a_r P_{ka} + \omega f^i_k f^a_r P_{ja} = \\ & = \bar{R}^b_{akj} \pi_{br} \tilde{\pi}^{ai} - \delta^i_r (\bar{P}_{kj} - \bar{P}_{jk}) - \omega f^i_r (f^a_j \bar{P}_{ka} - f^a_k \bar{P}_{ja}) - \\ & - \delta^i_j \bar{P}_{kr} + \delta^i_k \bar{P}_{jr} - \omega f^i_j f^a_r \bar{P}_{ka} + \omega f^i_k f^a_r \bar{P}_{ja}, \end{aligned}$$

or

$$\begin{aligned} & R^i_{rkj} - \delta^i_r (P_{kj} - P_{jk}) - \omega f^i_r (f^a_j P_{ka} - f^a_k P_{ja}) - \\ (4.13) \quad & - \pi_{jr} \tilde{\pi}^{ai} P_{ka} + \pi_{kr} \tilde{\pi}^{ai} P_{ja} - \omega f^a_j \pi_{ar} f^b_p \tilde{\pi}^{pi} P_{kb} + \omega f^a_k \pi_{ar} f^b_p \tilde{\pi}^{pi} P_{jb} = \\ & = \bar{R}^i_{rkj} - \delta^i_r (\bar{P}_{kj} - \bar{P}_{jk}) - \omega f^i_r (f^a_j \bar{P}_{ka} - f^a_k \bar{P}_{ja}) - \\ & - \pi_{jr} \tilde{\pi}^{ai} \bar{P}_{ka} + \pi_{kr} \tilde{\pi}^{ai} \bar{P}_{ja} - \omega f^a_j \pi_{ar} f^b_p \tilde{\pi}^{pi} \bar{P}_{kb} + \omega f^a_k \pi_{ar} f^b_p \tilde{\pi}^{pi} \bar{P}_{jb}. \end{aligned}$$

As for the tensors P_{rk} and $\bar{P}_{rk}$ they can be, taking into account (4.10) and (4.12), expressed as follows

$$P_{kr} = \frac{1}{2(n+1)} \left[{}^*R^b_k \pi_{br} + \frac{1}{n-1} ({}^*R^s_a \pi_{sb} + {}^*R^s_b \pi_{sa}) O_{rk}^{ab} \right],$$

$$\bar{P}_{kr} = \frac{1}{2(n+1)} \left[{}^*\bar{R}^b_k \pi_{br} + \frac{1}{n-1} ({}^*\bar{R}^s_a \pi_{sb} + {}^*\bar{R}^s_b \pi_{sa}) O_{rk}^{ab} \right].$$

Now, it can be seen at once that (4.13) is the same relation as (4.7).

References

- [1] S. ISHIHARA, Holomorphically projective changes and their groups in an almost complex manifold, *Tôhoku Math. J.* **9** (1959), 273—297.
- [2] T. OTSUKI and Y. TASHIRO, On curves in Kählerian spaces, *Math. J. Okayama Univ.* **4** (1954), 57—78.
- [3] Y. TASHIRO, On holomorphically projective correspondences in an almost complex space, *Math. J. Okayama Univ.* **6** (1957), 147—152.
- [4] K. YANO, Differential geometry on complex and almost complex spaces, *Pergamon Press*, 1965.
- [5] M. PRVANOVIĆ, Holomorphically projective transformations in a locally product space, *Mathematica Balkanica* **1** (1971), 195—213.
- [6] Г. Марков—А. П. Норден, О голоморфно—проективных преобразованиях, *Изв. вузов, Матем.*, 1975, Но. 6, 82—87.
- [7] K. RADZISZEWSKI, π -Geodesics and lines of shadow, *Colloquium Mathematicum* **26** (1972), 157—163.
- [8] A. BUCKI, π -Geodesics on hypersurfaces, *Ann. Univ. Mariae Curie-Skłodowska, Sec. A* **33** (1979), 29—44.

(Received April 22, 1987)