

On realizations of Fourier series operators

By JOUKO TERVO (Jyväskylä)

Abstract. The paper deals with linear operators $L(x, D)$ defined by

$$(1) \quad (L(x, D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)},$$

where φ lies in the space C_π^∞ of all smooth periodic functions, φ_l is the Fourier coefficient of φ and $L(\cdot, \cdot)$ is a mapping $\mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ such that $L(\cdot, l) \in C_\pi^\infty$ and that

$$(2) \quad \sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha (1 + |l|^2)^{\mu_\alpha/2}.$$

Here W is the cube $\{x \in \mathbb{R}^n | x_j \in]-\pi, \pi[\}$. The assumption (2) guarantees that $L(x, D)$ maps C_π^∞ continuously into C_π^∞ .

In the case when $\mu_\alpha = \mu + \delta|\alpha|$ with $\delta < 1$, the formal transpose $L'(x, D)$. $C_\pi^\infty \rightarrow C_\pi^\infty$ of $L(x, D)$ exists. Denote by $L_{p, k}^\sim$ (and $L_{p, k}^{\#}$) the minimal realization (the maximal realization, resp.) of $L(x, D)$ in the appropriate weighted spaces $\mathcal{B}_{p, k}^\pi$. Conditions for the bijectivity of $L_{p, k}^\sim + aI$ and for the essential maximality of $L(x, D)$, that is, for the equality of operators $L_{p, k}^\sim$ and $L_{p, k}^{\#}$ are established.

1. Introduction

Let $L(x, D)$ be a linear operator defined on the space C_π^∞ of all smooth periodic functions $\varphi: \mathbb{R}^n \rightarrow \mathbb{C}$ by the requirement

$$(1.1) \quad (L(x, D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)}.$$

Here $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ is a mapping such that $L(\cdot, l) \in C_\pi^\infty$ for $l \in \mathbb{Z}^n$ and that with some constants $C_\alpha > 0$ and $\mu_\alpha \in \mathbb{R}$ one has

$$(1.2) \quad \sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha (1 + |l|^2)^{\mu_\alpha/2} =: C_\alpha k_{\mu_\alpha}(l) \quad \text{for } l \in \mathbb{Z}^n.$$

The constant $\varphi_l := \int_W \varphi(x) e^{-i(l, x)} dx$ is the Fourier coefficient of φ . Every linear partial differential operator with C_π^∞ -coefficients can be presented in the form (1.1). For the theory of periodic partial differential operators cf. [3], [4], [2], [1] and [8]. In [11] it is showed some boundedness and compactness criteria for the operators (1.1) in the appropriate spaces and in [10] one-sided invertibility of the operators (1.1) is considered. For the related topic we also refer to [7].

The operators (1.1) for which (1.2) is valid form an algebra (cf. Corollary 2.2). In the case when μ_α is of the form $\mu_\alpha = \mu + \delta|\alpha|$ with $\delta < 1$, the formal transpose

$L'(x, D)$ of $L(x, D)$ exists and $L'(x, D)$ is of the form (1.1) (cf. Remark 2.6 and Theorem 2.8). Supposing that $L'(x, D)$ exists we can construct the minimal and maximal realizations, say $L_{p,k}^{\sim}$ and $L_{p,k}^{\#}$, of $L(x, D)$ in the weighted spaces $\mathcal{B}_{p,k}^{\pi}$ (cf. Section 2.4). In the case when $L(D) := L(x, D)$ is a translation invariant operator, one has $L_{p,k}^{\sim} = L_{p,k}^{\#}$ (cf. Theorem 3.1). The spectrum $\sigma(L_{p,k}^{\sim})$ of the minimal realization of the translation invariant operator $L(D)$ is given by $\sigma(L_{p,k}^{\sim}) = \overline{\{L(l) | l \in \mathbb{Z}^n\}}$ (cf. Theorem 3.4). This fact implies a criterion for the bijectivity of $L_{p,k}^{\sim} + aI$ when a is large enough.

In the fourth chapter we perturbate the translation invariant operator and obtain a condition for the bijectivity of $L_{p,k}^{\sim} + aI$, where $L_{p,k}^{\sim}$ is the minimal realization of the perturbed operator $L(x, D)$. A condition for the essential maximality, that is, for the equality $L_{p,k}^{\sim} = L_{p,k}^{\#}$, is also established. The fifth chapter considers operators (1.1), which are semi-bounded (from below) in appropriate spaces. Criteria for the bijectivity of $L_{p,k}^{\sim} + aI$ and for the equality $L_{p,k}^{\sim} = L_{p,k}^{\#}$ are proved. Our theory especially implies that for elliptic periodic partial differential operators one has $L_{2,k_t}^{\sim} = L_{2,k_t}^{\#}$, $N(L_{2,k_t}^{\#} + aI) = \{0\}$ and that $R(L_{2,k_t}^{\sim} + aI) = \mathcal{B}_{2,k_t}^{\pi}$ for a large enough, where t lies in $\mathbb{Z}$.

2. The algebra of Fourier series operators

2.1. Denote by C_{π}^{∞} the linear space of all $C^{\infty}(\mathbb{R}^n)$ -functions φ such that each function $x_j \rightarrow (x_1, \dots, x_j, \dots, x_n)$ is 2π -periodic. Equip the space C_{π}^{∞} with the (standard Frechet space) topology defined by the semi-norms $\varphi \rightarrow \sup_{x \in W} |(D^{\sigma} \varphi)(x)|$, $\sigma \in \mathbb{N}_0^n$. Here W denotes the cube $W := \{x \in \mathbb{R}^n | x_j \in]-\pi, \pi[, j=1, \dots, n\}$. The dual of C_{π}^{∞} is denoted by D'_{π} , in other words, D'_{π} is the linear space of all periodic distributions. We recall that for $T \in D'_{\pi}$ and for $\varphi \in C_{\pi}^{\infty}$ one has

$$(2.1) \quad T(\varphi) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} T_l \varphi_{-l},$$

where $T_l \in \mathbb{C}$ is defined by

$$(2.2) \quad T_l := T(e^{-i(l, \cdot)}),$$

and where $\varphi_l \in \mathbb{C}$ is the Fourier coefficient of φ , that is,

$$(2.3) \quad \varphi_l := \int_W \varphi(x) e^{-i(l, x)} dx =: \varphi(e^{-i(l, \cdot)}).$$

Furthermore, for each $T \in D'_{\pi}$ there exist constants $C > 0$ and $s \in \mathbb{R}$ such that

$$(2.4) \quad |T_l| \leq C(1 + |l|^2)^{s/2} =: Ck_s(l) \quad \text{for all } l \in \mathbb{Z}^n.$$

A distribution $T \in D'_{\pi}$ lies in C_{π}^{∞} if and only if for each $N \in \mathbb{N}$ there exists a constant $C > 0$ such that

$$(2.5) \quad |T_l| \leq Ck_{-N}(l) \quad \text{for all } l \in \mathbb{Z}^n.$$

The product $\psi T \in D'_{\pi}$ of $T \in D'_{\pi}$ and $\psi \in C_{\pi}^{\infty}$ is defined through the relation

$$(2.6) \quad (\psi T)(\varphi) = T(\psi \varphi) \quad \text{for } \varphi \in C_{\pi}^{\infty}.$$

Then one has

$$(2.7) \quad (\psi T)_l = (2\pi)^{-n} \sum_{z \in \mathbb{Z}^n} T_z \psi_{l-z}.$$

For the definition of Banach spaces $\mathcal{B}_{p,k}^\pi$ we refer to [11]. Here p lies in the interval $[1, \infty[$ and k belongs to an appropriate class $\mathcal{K}_\pi$ of weight functions $k: \mathbb{Z}^n \rightarrow \mathbb{R}$. We have the topological inclusions $C_\pi^\infty \subset \mathcal{B}_{p,k}^\pi \subset D'_\pi$, when D'_π is equipped with the weak dual topology. The relations

$$C_\pi^\infty = \bigcap_{k \in \mathcal{K}_\pi} \mathcal{B}_{p,k}^\pi$$

and

$$D'_\pi = \bigcup_{k \in \mathcal{K}_\pi} \mathcal{B}_{p,k}^\pi$$

are also valid. Let $p' \in [1, \infty]$ such that $1/p + 1/p' = 1$ and let $k^\vee \in \mathcal{K}_\pi$ such that $k^\vee(l) = k(-l)$. Denote by $\mathcal{B}_{p,k}^{\pi*}$, $p \in [1, \infty[$ the dual space of $\mathcal{B}_{p,k}^\pi$. Then for each $t \in \mathcal{B}_{p,k}^{\pi*}$ there exists a unique $T \in \mathcal{B}_{p',1/k^\vee}^\pi$ such that

$$(2.1) \quad t\varphi = T(\varphi) \quad \text{for } \varphi \in C_\pi^\infty.$$

On the other hand, the linear form

$$(2.9) \quad \varphi \rightarrow T(\varphi), \quad T \in \mathcal{B}_{p',1/k^\vee}^\pi$$

has a unique continuous extension from $\mathcal{B}_{p,k}^\pi$ to $\mathbb{C}$. Furthermore, the mapping $\mathcal{L}: \mathcal{B}_{p',1/k^\vee}^{\pi*} \rightarrow \mathcal{B}_{p,k}^{\pi*}$ defined by

$$(2.10) \quad \ell(T) = t$$

is an isometrical isomorphism. For some additional properties of $\mathcal{B}_{p,k}^\pi$ -spaces we refer to [11], [10] and [7].

2.2. In the sequel $L(\cdot, \cdot)$ denotes a mapping $\mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ such that

$$(2.11) \quad L(\cdot, l) \in C_\pi^\infty \quad \text{for each } l \in \mathbb{Z}^n.$$

We shall deal with linear operators $L(x, D)$ defined by the requirement

$$(2.12) \quad (L(x, D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(x, l) \varphi_l e^{i(l, x)} \quad \text{for } \varphi \in C_\pi^\infty.$$

Supposing that with some $C > 0$ and $M \in \mathbb{R}$ one has

$$(2.13) \quad \sup_{x \in W} |L(x, l)| \leq C k_M(l) \quad \text{for all } l \in \mathbb{Z}^n$$

one sees that $L(x, D)\varphi$ is a mapping $\mathbb{R}^n \rightarrow \mathbb{C}$, which, in addition is 2π -periodic with respect to each variable.

Furthermore, we have

Theorem 2.1. Suppose that for each $\alpha \in \mathbb{N}_0^n$ there exist constants $C_\alpha > 0$ and $\mu_\alpha \in \mathbb{R}$ such that

$$(2.14) \quad \sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu_\alpha}(l) \quad \text{for all } l \in \mathbb{Z}^n.$$

Then the operator $L(x, D)$ defined by (2.12) maps C_π^∞ continuously into C_π^∞ .

On the contrary, suppose that a linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ is continuous. Then there exists a mapping $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ such that (2.11) and (2.14) are valid and that one has

$$(2.15) \quad L = L(x, D).$$

PROOF. A) The proof of the first part of the assertion is established in [11], Theorem 3.1; cf. also [10].

B) We show the last part of the assertion (for an alternative proof cf. also [10]). Let L be a continuous operator $C_\pi^\infty \rightarrow C_\pi^\infty$. Then for each $\alpha \in \mathbb{N}_0^n$ there exist constants $C_\alpha > 0$ and $N_\alpha \in \mathbb{N}$ such that

$$(2.16) \quad \sup_{x \in W} |D^\alpha(L\varphi)(x)| \leq C_\alpha \sum_{|\beta| \leq N_\alpha} \sup_{x \in W} |(D^\beta \varphi)(x)| \quad \text{for all } \varphi \in C_\pi^\infty$$

(cf. [12], p. 42). Define a mapping $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ by

$$(2.17) \quad L(x, l) = [L(e^{i(l, \cdot)})](x) e^{-i(l, x)}.$$

Then $L(\cdot, l)$ belongs to C_π^∞ for each $l \in \mathbb{Z}^n$ and by (2.16) one has

$$\begin{aligned} (2.18) \quad |D_x^\alpha L(x, l)| &= \left| \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^{x-\gamma}(L(e^{i(l, \cdot)}))(x) D^\gamma(e^{-i(l, \cdot)})(x) \right| \leq \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_{\alpha-\gamma} \sum_{|\beta| \leq N_{\alpha-\gamma}} \sup_{x \in W} |D^\beta(e^{i(l, \cdot)})(x)| |(-l)^\gamma e^{i(l, x)}| \leq \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_{\alpha-\gamma} \sum_{|\beta| \leq N_{\alpha-\gamma}} |l^{\beta+\gamma}| \leq \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} C_{\alpha-\gamma} \sum_{|\beta| \leq N_{\alpha-\gamma}} k_{N_{\alpha-\gamma}+|\gamma|}(l) \leq C'_\alpha k_{\mu_\alpha}(l) \end{aligned}$$

for all $l \in \mathbb{Z}^n$ and $x \in W$, where C'_α is a suitable constant and where $\mu_\alpha := \max_{\gamma \leq \alpha} \{N_{\alpha-\gamma}\} + |\alpha|$. Hence the estimate (2.14) is valid.

We have to show that $L = L(x, D)$. Let $\varphi \in C_\pi^\infty$ and let $\varphi_n \in C_\pi^\infty$ such that

$$\varphi_n(x) = (2\pi)^{-n} \sum_{|l| \leq n} \varphi_l e^{i(l, x)}$$

(that is, $(\varphi_n)_l = \varphi_l$ for $|l| \leq n$ and $(\varphi_n)_l = 0$ for $|l| > n$). Then one sees that $\varphi_n \rightarrow \varphi$ in C_π^∞ . Hence the continuity of L implies that

$$(2.19) \quad L\varphi_n \rightarrow L\varphi \quad \text{in } C_\pi^\infty.$$

Let $L(x, D)$ be the operator (2.12) with the symbol (2.17). In virtue of the first part of the assertion $L(x, D): C_\pi^\infty \rightarrow C_\pi^\infty$ is continuous, as well. Hence one has

$$(2.20) \quad L(x, D)\varphi_n \rightarrow L(x, D)\varphi \quad \text{in } C_\pi^\infty.$$

Noting that

$$\begin{aligned} (2.21) \quad (L\varphi_n)(x) &= [(2\pi)^{-n} \sum_{|l| \leq n} \varphi_l L(e^{i(l, \cdot)})](x) = \\ &= (2\pi)^{-n} \sum_{|l| \leq n} ([L(e^{i(l, \cdot)})](x) e^{-i(l, x)}) \varphi_l e^{i(l, x)} = (L(x, D)\varphi_n)(x), \end{aligned}$$

one obtains $L\varphi_n = L(x, D)\varphi_n$. Hence by (2.19)–(2.20) one has $L\varphi = L(x, D)\varphi$, as required. $\square$

Denote by S^π the linear space of all mappings $L(\cdot, \cdot): \mathbb{R}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ such that $L(\cdot, l) \in C_\pi^\infty$ for each $l \in \mathbb{Z}^n$ and that

$$(2.14) \quad \sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu_\alpha}(l) \quad \text{for all } l \in \mathbb{Z}^n.$$

In virtue of Theorem 2.1 for each $L(\cdot, \cdot) \in S^\pi$ there exists the continuous linear operator $L(x, D): C_\pi^\infty \rightarrow C_\pi^\infty$ defined by (2.12) and, in vice versa, every continuous linear operator $L: C_\pi^\infty \rightarrow C_\pi^\infty$ is of the form $L(x, D)$ with $L(\cdot, \cdot) \in S^\pi$. We denote by L^π the linear space $\{L(x, D) | L(\cdot, \cdot) \in S^\pi\}$ of linear operators $C_\pi^\infty \rightarrow C_\pi^\infty$. Then the mapping $\kappa: S^\pi \rightarrow L^\pi$ such that

$$\kappa(L(\cdot, \cdot)) = L(x, D)$$

is a well-defined linear bijection.

Let $\circ$ denote the composition of two operators $L(x, D)$ and $K(x, D) \in L^\pi$. Since (due to Theorem 2.1) L^π is exactly the linear space of all continuous linear operators $C_\pi^\infty \rightarrow C_\pi^\infty$ one has (for the definition of an algebra cf. [5], p. 140, for example)

Corollary 2.2. *The pair $(L^\pi, \circ)$ forms an algebra with unit.* $\square$

Remark 2.3. Since one has

$$(2.22) \quad \varphi(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} \varphi_l e^{i(l, x)},$$

the symbol $I(\cdot, \cdot)$ of the identical operator (which is the unit of the algebra $(L^\pi, \circ)$) is defined by $I(x, l) = 1$.

For the symbol $(L \circ K)(\cdot, \cdot)$ of the composed operator $L(x, D) \circ K(x, D)$ we obtain

Theorem 2.4. *Let $L(\cdot, \cdot)$ and $K(\cdot, \cdot)$ be in S^π . Then $(L \circ K)(\cdot, \cdot)$ has the expression*

$$(2.23) \quad (L \circ K)(x, l) = (2\pi)^{-n} \sum_{z \in \mathbb{Z}^n} L(x, z) (K(\cdot, l))_{l-z} e^{i(l-z, x)}.$$

PROOF. A) At first we verify that $(L \circ K)(\cdot, \cdot)$ lies in S^π . For each $\gamma \in \mathbb{N}_0^n$ one has with suitable constants $C_\gamma > 0$ and $\mu_\gamma \in \mathbb{R}$

$$(2.24) \quad \begin{aligned} |(l-z)^\gamma (K(\cdot, l))_{l-z}| &= |(D_x^\gamma K(\cdot, l))_{l-z}| \leq \\ &\leq \sup_{x \in W} |D_x^\gamma K(x, l)| \int_W 1 \, dx = (2\pi)^n C'_\gamma k_{\mu'_\gamma}(l) \quad \text{for } l, z \in \mathbb{Z}^n. \end{aligned}$$

Here we used the fact that by the periodicity the relation

$$(2.25) \quad l^\gamma \varphi_l = (D^\gamma \varphi)_l$$

holds. Due to (2.24) for each $m \in \mathbb{N}$ there exist constants $C_m > 0$ and $\mu_m \in \mathbb{R}$ such that

$$(2.26) \quad |(K(\cdot, l))_{l-z}| \leq C_m k_{\mu_m}(l) k_{-m}(z).$$

For each $\alpha \in \mathbb{N}_0^n$ we get

$$\begin{aligned}
 (2.27) \quad |D_x^\alpha(L(\cdot, z)e^{i(l-z, \cdot)})(x)| &\leq \sum_{v \leq \alpha} \binom{\alpha}{v} |D_x^{\alpha-v} L(x, z)| |(l-z)^v| \leq \\
 &\leq \sum_{v \leq \alpha} \binom{\alpha}{v} C_{\alpha-v} k_{\mu_{\alpha-v}}(z) k_{|v|}(l-z) \leq \\
 &\leq C_\alpha k_{\mu_\alpha + |\alpha|}(z) k_{|\alpha|}(l),
 \end{aligned}$$

where C_α is a suitable constant and where $\mu_\alpha := \max_{v \leq \alpha} \{\mu_{\alpha-v}\}$. Choosing in (2.26) $m = n+1 + \mu_\alpha + |\alpha|$ one sees by (2.27) that the series

$$\sum_{z \in \mathbb{Z}^n} D_x^\alpha(L(\cdot, z)e^{i(l, \cdot)})(x)(K(\cdot, l))_{l-z}$$

is absolutely and uniformly convergent in $\mathbb{R}^n$. Hence $D_x^\alpha(L \circ K)(x, l)$ exists, $D_x^\alpha(L \circ K)(\cdot, l)$ is continuous and

$$(2.28) \quad D_x^\alpha(L \circ K)(x, l) = (2\pi)^{-n} \sum_{z \in \mathbb{Z}^n} D_x^\alpha(L(\cdot, z)e^{i(l, \cdot)})(x)(K(\cdot, l))_{l-z}.$$

Thus $(L \circ K)(\cdot, l) \in C_\pi^\infty$ and by (2.26)–(2.27) one has

$$\sup_{x \in W} |D_x^\alpha(L \circ K)(x, l)| \leq C_\alpha C_m \left(\sum_{z \in \mathbb{Z}^n} k_{-(n+1)}(z) \right) k_{|\alpha| + \mu_m}(l)$$

where we chose $m = n+1 + \mu_\alpha + |\alpha|$. Hence $(L \circ K)(\cdot, \cdot) \in S^\pi$.

B) Let $\varphi \in C_\pi^\infty$ and let $\varphi_n \in C_\pi^\infty$ be defined by $\varphi_n(x) = (2\pi)^{-n} \sum_{|l| \leq n} \varphi_l e^{i(l, x)}$. Since $\varphi_n \rightarrow \varphi$ in C_π^∞ one has

$$(2.29) \quad (L(x, D) \circ K(x, D))\varphi_n \rightarrow L(x, D) \circ K(x, D)\varphi$$

and

$$(2.30) \quad ((L \circ K)(x, D))\varphi_n \rightarrow (L \circ K)(x, D)\varphi.$$

To complete the proof it suffices to note that

$$\begin{aligned}
 (2.31) \quad (L(x, D) \circ K(x, D))\varphi_n &= (2\pi)^{-n} L(x, D) \left(\sum_{|l| \leq n} K(\cdot, l) e^{i(l, \cdot)} \varphi_l \right) = \\
 &= (2\pi)^{-n} \sum_{|l| \leq n} [L(x, D)(K(\cdot, l) e^{i(l, \cdot)}) e^{-i(l, \cdot)}] \varphi_l e^{i(l, \cdot)} = \\
 &= (2\pi)^{-2n} \sum_{|l| \leq n} \left[\sum_{z \in \mathbb{Z}^n} L(x, z) (K(\cdot, l) e^{i(l, \cdot)})_z e^{i(z-l, \cdot)} \right] \varphi_l e^{i(l, \cdot)} = \\
 &= (2\pi)^{-2n} \sum_{|l| \leq n} \left[\sum_{z \in \mathbb{Z}^n} L(x, z) (K(\cdot, l))_{z-l} e^{i(z-l, \cdot)} \right] \varphi_l e^{i(l, \cdot)} = \\
 &= (L \circ K)\varphi_n. \quad \square
 \end{aligned}$$

Remark 2.5. A) The symbol $(L \circ K)(\cdot, \cdot)$ can also be expressed in the form

$$(2.32) \quad (L \circ K)(x, l) = (2\pi)^{-n} \sum_{z \in \mathbb{Z}^n} \left(\int_W L(x, z) K(y, l) e^{i(z-l, x-y)} dy \right).$$

B) Let S_δ^π ; $\delta \in \mathbf{R}$ be the linear subspace of S^π such that $L(\cdot, \cdot) \in S_\delta^\pi$ if the estimate

$$\sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu+\delta|\alpha|}(l) \quad \text{for all } l \in \mathbf{Z}^n$$

holds. Equip the space S_δ^π with a locally convex topology defined by the semi-norms

$$L(\cdot, \cdot) \rightarrow \sup_{l \in \mathbf{Z}^n} \left(\sup_{x \in W} |D_x^\alpha L(x, l)| \right) / k_{\mu+\delta|\alpha|}(l), \quad \alpha \in \mathbf{N}_0^n.$$

Since C_π^∞ is complete, one sees that S_δ^π is a Frechet space.

2.3. Let L be a linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$. We say that the formal transpose of L exists, if one is able to find a linear operator $L': C_\pi^\infty \rightarrow C_\pi^\infty$ such that

$$(2.33) \quad (L\varphi)(\psi) := \int_W (L\varphi)(x) \psi(x) dx = \varphi(L'\psi) \quad \text{for all } \varphi, \psi \in C_\pi^\infty.$$

The operator L' is the formal transpose of L . One sees that L' is unique and that $(L')' = L$.

Remark 2.6. Suppose that for each $\alpha \in \mathbf{N}_0^n$ there exists a constant $C_\alpha > 0$ such that

$$(2.34) \quad \sup_{x \in W} |D_x^\alpha L(x, l)| \leq C_\alpha k_{\mu+\delta|\alpha|}(l) \quad \text{for } l \in \mathbf{Z}^n,$$

where $\mu \in \mathbf{R}$ and where $\delta < 1$. Then the formal transpose $L'(x, D): C_\pi^\infty \rightarrow C_\pi^\infty$ of $L(x, D)$ exists (cf. [11]).

Lemma 2.7. Let L be a linear operator $C_\pi^\infty \rightarrow C_\pi^\infty$ such that the formal transpose $L': C_\pi^\infty \rightarrow C_\pi^\infty$ exists. Then L (and L') is continuous.

PROOF. Since C_π^∞ is a Frechet space we have only to verify that $L: C_\pi^\infty \rightarrow C_\pi^\infty$ is closed (cf. the Closed Graph Theorem given e.g. in [12], p. 79). Let $\{\varphi_n\} \subset C_\pi^\infty$ be a sequence such that $\varphi_n \rightarrow \varphi$ and $L\varphi_n \rightarrow \psi$ in C_π^∞ with some $\varphi, \psi \in C_\pi^\infty$. Then one has for each $l \in \mathbf{N}_0^n$

$$(2.35) \quad \begin{aligned} \psi_l &= \psi(e^{-i(l, \cdot)}) = \lim_{n \rightarrow \infty} (L\varphi_n)(e^{-i(l, \cdot)}) \\ &= \lim_{n \rightarrow \infty} \varphi_n(L'(e^{-i(l, \cdot)})) = \varphi(L'(e^{-i(l, \cdot)})) = (L\varphi)_l. \end{aligned}$$

This shows that $L\varphi = \psi$ and then L is closed. $\square$

Denote by $L^{\pi\pi}$ the linear subspace of L^π defined by

$$(2.36) \quad L^{\pi\pi} = \{L \in L^\pi \mid \text{the formal transpose } L' \text{ exists}\}.$$

If $L \in L^{\pi\pi}$ then $L' \in L^{\pi\pi}$ (cf. Lemma 2.7). Furthermore, one has

Theorem 2.8. The pair $(L^{\pi\pi}, \circ)$ forms a subalgebra (with unit) of $(L^\pi, \circ)$.

PROOF. Let $L = L(x, D)$ and $K = K(x, D)$ be in $L^{\pi\pi}$. Then one has

$$(2.37) \quad ((L \circ K)\varphi)(\psi) = (K\varphi)(L'\psi) = \varphi((K' \circ L')\psi) \quad \text{for } \varphi, \psi \in C_\pi^\infty.$$

Hence $(L \circ K)'$ exists and $(L \circ K)' = K' \circ L' \in L^\pi$. Clearly I' exists and $I' = I \in L^\pi$. This completes the proof. $\square$

Let $S^{\pi\pi}$ be a subclass of S^π defined by $S^{\pi\pi} = \kappa^{-1}(L^{\pi\pi})$. We have

Theorem 2.9. *Let $L(\cdot, \cdot)$ be in $S^{\pi\pi}$. Then the symbol $L'(\cdot, \cdot)$ of $L'(x, D)$ has the expression*

$$(2.38) \quad L'(x, z) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} (L(\cdot, -(z+l)))_l e^{i(l, x)} \quad \text{for } z \in \mathbb{Z}^n, x \in \mathbb{R}^n.$$

PROOF. Since $L'(x, D) \in L^\pi$ there exists $L'(\cdot, \cdot) \in S^\pi$ such that

$$(L'(x, D)\psi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L'(x, l) \psi_l e^{i(l, x)}$$

and the one has

$$(2.39) \quad L'(x, z) = [L'(x, D)(e^{i(z, \cdot)})](x) e^{-i(z, x)} \quad \text{for } z \in \mathbb{Z}^n.$$

Furthermore, we obtain for each $z \in \mathbb{Z}^n$

$$\begin{aligned} (L'(\cdot, z))_l &= (L'(\cdot, z))(e^{-i(l, \cdot)}) = \\ &= ([L'(x, D)(e^{i(z, \cdot)})]e^{-i(l, \cdot)})(e^{-i(l, \cdot)}) = \\ &= (e^{i(z, \cdot)})(L(x, D)(e^{-i(z+l, \cdot)})) = (L(\cdot, -(z+l)))_l \end{aligned}$$

and then one has

$$\begin{aligned} L'(x, z) &= (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} (L'(\cdot, z))_l e^{i(l, x)} = \\ &= (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} (L(\cdot, -(z+l)))_l e^{i(l, x)}, \end{aligned}$$

as required. $\square$

Remark 2.10. In virtue of (2.40) the symbols $L(\cdot, \cdot)$ and $L'(\cdot, \cdot)$ satisfy the relation

$$(2.40) \quad (L'(\cdot, z))_l = (L(\cdot, -(z+l)))_l \quad \text{for } z, l \in \mathbb{Z}^n.$$

2.4. Let $L = L(x, D)$ be in $L^{\pi\pi}$. Then we are able to construct the linear operators $L_{p,k}^{\#}$ and $L_{p,k}: \mathcal{B}_{p,k}^\pi \rightarrow \mathcal{B}_{p,k}^\pi$ as follows

$$(2.41) \quad \begin{cases} D(L_{p,k}^{\#}) = \{u \in \mathcal{B}_{p,k}^\pi \mid \exists f \in \mathcal{B}_{p,k}^\pi \text{ such that } u(L'\varphi) = f(\varphi) \text{ for } \varphi \in C_\pi^\infty\} \\ L_{p,k}^{\#} u = f, \end{cases}$$

and

$$(2.42) \quad \begin{cases} D(L_{p,k}) = C_\pi^\infty \\ L_{p,k}\varphi = L\varphi \quad \text{for } \varphi \in C_\pi^\infty. \end{cases}$$

One sees that $L_{p,k}^{\#}$ and $L_{p,k}$ are densely defined. Furthermore, $L_{p,k}^{\#}$ is closed and $L_{p,k}$ is closable (cf. [12], pp. 77 and use (2.6) of [11]). Let $\tilde{L}_{p,k}: \mathcal{B}_{p,k}^\pi \rightarrow \mathcal{B}_{p,k}^\pi$ be the smallest closed extension of $L_{p,k}$. One always has $\tilde{L}_{p,k} \subset L_{p,k}^{\#}$, that is, $L_{p,k}^{\#}$ is the extension of $\tilde{L}_{p,k}$.

3. The translation invariant operator $L(D)$

3.1. Let $L(\cdot): \mathbb{Z}^n \rightarrow \mathbb{C}$ be a mapping such that

$$(3.1) \quad |L(l)| \leq C k_\mu(l) \quad \text{for } l \in \mathbb{Z}^n.$$

Then by Theorem 2.1, by Remark 2.6 and by Theorem 2.9 the operator $L(D)$ defined by

$$(3.2) \quad (L(D)\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} L(l) \varphi_l e^{i(l, x)}$$

maps C_π^∞ continuously into C_π^∞ , the formal transpose $L'(D)$ of $L(D)$ exists and $L'(1) = L(-1)$. In addition, we have for $p \in [1, \infty]$

Theorem 3.1. Suppose that $L(\cdot)$ obeys (3.1). Then the identity

$$(3.3) \quad L_{p,k}^\sim = L_{p,k}^{\prime\sharp}$$

holds.

PROOF. Let u be in $D(L_{p,k}^{\prime\sharp})$ and let $L_{p,k}^{\prime\sharp} u = f$. Then for each $l \in \mathbb{Z}^n$ we get

$$(3.4) \quad \begin{aligned} f_l &= (L_{p,k}^{\prime\sharp} u)(e^{-i(l, \cdot)}) = u(L'(D)(e^{-i(l, \cdot)})) \\ &= u(L(l)e^{-i(l, \cdot)}) = L(l)u_l. \end{aligned}$$

Let $\{u_n\} \subset C_\pi^\infty$ be a sequence such that

$$u_n(x) = (2\pi)^{-n} \sum_{|l| \leq n} u_l e^{i(l, x)}.$$

Let $p \in [1, \infty[$. Then one sees that $\|u_n - u\|_{p,k} = ((2\pi)^{-n} \sum_{|l| > n} |u_l k(l)|^p)^{1/p} \rightarrow 0$ and that by (3.4)

$$\|L_{p,k}^{\prime\sharp} u_n - f\|_{p,k} = ((2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} |(L(l)u_n)_l - f_l| k(l)|^p)^{1/p} = ((2\pi)^{-n} \sum_{|l| > n} |f_l k(l)|^p)^{1/p} \rightarrow 0$$

with $n \rightarrow \infty$. Hence $u \in D(L_{p,k}^\sim)$ and $L_{p,k}^\sim u = f$, as required. $\square$

3.2. Let p be in the interval $[1, \infty[$ and let k and $k^\sim$ be in $\mathcal{K}_\pi$. We have

Theorem 3.2. Suppose that $L(\cdot)$ obeys (3.1). Then the inequality

$$(3.5) \quad \|L(D)\varphi\|_{p,k} \leq C_1 \|\varphi\|_{p,kk^\sim} - C_2 \|\varphi\|_{p,k} \quad \text{for all } \varphi \in C_\pi^\infty$$

holds with $C_1 > 0$ and $C_2 \geq 0$ if and only if there exists a constant $C > 0$ such that

$$(3.6) \quad k^\sim(l) \leq C(1 + |L(l)|) \quad \text{for all } l \in \mathbb{Z}^n.$$

PROOF. Suppose that (3.5) holds. For each $k \in \mathcal{K}_\pi$, $p \in [1, \infty[$ and $l \in \mathbb{Z}^n$ one has

$$(3.7) \quad \begin{aligned} \|L(D)(e^{i(l, \cdot)})\|_{p,k} &= ((2\pi)^{-n} \sum_{z \in \mathbb{Z}^n} |(L(D)(e^{i(l, \cdot)}))_z k(z)|^p)^{1/p} = \\ &= ((2\pi)^{-n} \sum_{z \in \mathbb{Z}^n} |L(l)(e^{i(l, \cdot)})_z k(z)|^p)^{1/p} = |L(l)| k(l) \end{aligned}$$

and

$$\|e^{l(\cdot, \cdot)}\|_{p,k} = k(l).$$

Hence (3.6) is valid.

Multiplying (3.6) by $|\varphi_l k(l)|$ and taking into account that $(L(D)\varphi)_l = L(l)\varphi_l$ one sees easily that (3.6) implies (3.5). $\square$

Suppose that $p=2$. Then the space $\mathcal{H}_k^\pi := \mathcal{B}_{2,k}^\pi$ is a Hilbert space with the inner product

$$(3.8) \quad (u, v)_k = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} u_l v_l k^2(l).$$

Similarly as Theorem 3.2 one has obviously

Theorem 3.3. *Suppose that $L(\cdot)$ obeys (3.1). Then the inequality*

$$(3.9) \quad \operatorname{Re}(L(D)\varphi, \varphi)_k \geq C_1 \|\varphi\|_{kk}^2 - C_2 \|\varphi\|_k^2 \quad \text{for } \varphi \in C_\pi^\infty$$

holds with $C_1 > 0$ and $C_2 \geq 0$ if and only if there exists a constant $C > 0$ such that

$$(3.10) \quad k^*(l) \leq C(\operatorname{Re} L(l) + 1) \quad \text{for } l \in \mathbb{Z}^n.$$

3.3. Denote by $\sigma(L_{p,k}^\sim)$ the spectrum of $L_{p,k}^\sim$. Let $L_{p,k}^{\sim*}: \mathcal{B}_{p,k}^{\pi*} \rightarrow \mathcal{B}_{p,k}^{\pi*}$ be the dual operator of $L_{p,k}^\sim$. Because $L_{p,k}^\sim$ is a closed operator, the scalar λ lies in the resolvent set $\varrho(L_{p,k}^\sim) := \mathbb{C} \setminus \sigma(L_{p,k}^\sim)$ of $L_{p,k}^\sim$ if and only if the relations

$$(3.11) \quad N(L_{p,k}^\sim - \lambda I) = \{0\}$$

and

$$(3.12) \quad R(L_{p,k}^\sim - \lambda I) = \mathcal{B}_{p,k}^\pi$$

hold (cf. [9], p. 14). We obtain for $p \in [1, \infty[$

Theorem 3.4. *Suppose that $L(\cdot)$ obeys (3.1). Then one has*

$$(3.13) \quad \sigma(L_{p,k}^\sim) = \{\overline{L(l)} \mid l \in \mathbb{Z}^n\}.$$

PROOF. A) Suppose that $\lambda \notin \sigma(L_{p,k}^\sim)$, that is, $\lambda \in \varrho(L_{p,k}^\sim)$. Then by (3.11)–(3.12) then inverse operator $(L_{p,k}^\sim - \lambda I)^{-1}: \mathcal{B}_{p,k}^\pi \rightarrow \mathcal{B}_{p,k}^\pi$ exists and $D((L_{p,k}^\sim - \lambda I)^{-1}) = \mathcal{B}_{p,k}^\pi$. Since the operator $(L_{p,k}^\sim - \lambda I)^{-1}$ is closed, the Closed Graph Theorem implies that $(L_{p,k}^\sim - \lambda I)^{-1}$ is bounded. Hence one has

$$(3.14) \quad \|u\|_{p,k} \leq \|(L_{p,k}^\sim - \lambda I)^{-1}\| \|(L_{p,k}^\sim - \lambda I)u\|_{p,k} \quad \text{for } u \in D(L_{p,k}^\sim).$$

Choosing $u = e^{l(\cdot, \cdot)}$ one obtains that

$$(3.15) \quad |L(l) - \lambda| \geq 1/\|(L_{p,k}^\sim - \lambda I)^{-1}\| \quad \text{for all } l \in \mathbb{Z}^n,$$

and then $\lambda \notin \{\overline{L(l)} \mid l \in \mathbb{Z}^n\}$.

B) Conversely, suppose that $\lambda \in \{\overline{L(l)} \mid l \in \mathbb{Z}^n\}$. Then there exists a constant $c > 0$ such that

$$(3.16) \quad |L(l) - \lambda| \geq c \quad \text{for all } l \in \mathbb{Z}^n.$$

Hence one has for all $\varphi \in C_\pi^\infty$

$$(3.17) \quad \|(L(D) - \lambda I)\varphi\|_{p,k} \cong c \|\varphi\|_{p,k},$$

which implies that $R(L_{p,k}^\sim - \lambda I)$ is closed and that

$$(3.18) \quad N(L_{p,k}^\sim - \lambda I) = \{0\}.$$

We establish that

$$(3.19) \quad N(L_{p,k}^{\sim*} - \lambda I) = \{0\}.$$

Choose U from $N(L_{p,k}^{\sim*} - \lambda I) \subset \mathcal{B}_{p,k}^{\pi*}$. Let $u \in \mathcal{B}_{p',1/k}^\pi$ be defined by $u = \mathcal{L}^{-1}(U)$ (cf. Section 2.1). Then one has for all $\varphi \in C_\pi^\infty$

$$(3.20) \quad u((L(D) - \lambda I)\varphi) = U((L(D) - \lambda I)\varphi) = U((L_{p,k}^\sim - I)\varphi) = 0$$

and then $u \in N(L_{p',1/k}^\# - \lambda I)$. In virtue of Theorem 3.1 one has $L_{p',1/k}^\# = L_{p',1/k}^{\sim'}$. Due to (3.16) we also have

$$\|(L'(D) - \lambda I)\varphi\|_{p',1/k} \cong c \|\varphi\|_{p',1/k}$$

and then $N(L_{p',1/k}^\# - \lambda I) = N(L_{p',1/k}^{\sim'} - \lambda I) = \{0\}$. This shows that $U = \mathcal{L}(u) = 0$.

Since $R(L_{p,k}^\sim - \lambda I)$ is closed, the relation (3.19) implies that $R(L_{p,k}^\sim - \lambda I) = \mathcal{B}_{p,k}^\pi$. Hence $\lambda \in \varrho(L_{p,k}^\sim)$, that is, $\lambda \notin \sigma(L_{p,k}^\sim)$. This completes the proof. $\square$

Remark 3.5. Suppose that $|L(l)| \rightarrow \infty$ with $|l| \rightarrow \infty$. Then the set $\{L(l) | l \in \mathbb{Z}^n\}$ is closed and so one has

$$(3.21) \quad \sigma(L_{p,k}^\sim) = \{L(l) | l \in \mathbb{Z}^n\}.$$

For $p \in [1, \infty[$ we obtain

Corollary 3.6. *Suppose that $L(\cdot)$ obeys (3.1). Furthermore, suppose that there exists a positive constant $c > 0$ such that*

$$(3.22) \quad \operatorname{Re} L(l) \geq -c \quad \text{for all } l \in \mathbb{Z}^n.$$

Then the relations

$$(3.23) \quad N(L_{p,k}^\sim + aI) = \{0\} \quad \text{and} \quad R(L_{p,k}^\sim + aI) = \mathcal{B}_{p,k}^\pi$$

hold for $a > c$.

PROOF. In virtue of (3.22) one has

$$(3.24) \quad |L(l) + a| \geq \operatorname{Re} L(l) + a = a - c > 0 \quad \text{for all } l \in \mathbb{Z}^n.$$

Hence $-a$ does not belong to $\overline{\{L(l) | l \in \mathbb{Z}^n\}}$ and then by Theorem 3.4 $-a$ belongs to $\varrho(L_{p,k}^\sim)$. This proves the assertion. $\square$

Remark 3.7. A) The proof of Corollary 3.6 shows that the condition (with $\lambda \in \mathbb{C}$)

$$(3.25) \quad |L(l) + \lambda I| \cong E > 0 \quad \text{for all } l \in \mathbb{Z}^n$$

implies the relations (3.11)—(3.12).

B) The elements k of $\mathcal{K}_\pi$ satisfy (3.1). In the Chapter 5 we shall denote the corresponding operators (3.2) by $k(D)$.

4. The perturbed operator

In the sequel we assume that the operator $L = L(x, D)$ belongs to L^{t^n} and that $p \in [1, \infty[$. At first we establish

Theorem 4.1. *Suppose that $P(\cdot): \mathbb{Z}^n \rightarrow \mathbb{C}$ obeys the estimates*

$$(4.1) \quad |P(l)| \cong Ck_M(l)$$

and

$$(4.2) \quad \operatorname{Re} P(l) \cong -c \quad \text{for all } l \in \mathbb{Z}^n,$$

where C, M and c are positive constants. Furthermore, suppose that $L \in L^{t^n}$ such that

$$(4.3) \quad \|(L - P)\varphi\|_{p,k} \cong \alpha \|P\varphi\|_{p,k} + \beta \|\varphi\|_{p,k} \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $\alpha \in]0, 1[$, $\beta \cong 0$ and $P = P(D)$.

Then there exists $R \cong 0$ such that the relations

$$(4.4) \quad N(L_{p,k}^\sim + aI) = \{0\} \quad \text{and} \quad R(L_{p,k}^\sim + aI) = \mathcal{B}_{p,k}^\pi$$

hold for $a \cong R$.

PROOF. Due to Corollary 3.6 the operator $P_{p,k}^\sim + aI$ is a Fredholm operator with

$$(4.5) \quad \operatorname{ind}(P_{p,k}^\sim + aI) = 0 \quad \text{for } a > c.$$

In virtue of (4.2) we obtain

$$|P(l) + a| \cong a - c \quad \text{for } l \in \mathbb{Z}^n$$

and then one has

$$(4.6) \quad \|(P + aI)\varphi\|_{p,k} \cong (a - c)\|\varphi\|_{p,k} \quad \text{for all } \varphi \in C_\pi^\infty.$$

Furthermore, one has for $a \cong 2c$

$$|P(l) + a|^2 = |P(l)|^2 + 2a \operatorname{Re} P(l) - a^2 \cong$$

$$\cong |P(l)|^2 - 2ac + a^2 \cong |P(l)|^2 \quad \text{for all } l \in \mathbb{Z}^n$$

and then

$$(4.7) \quad \|(P + aI)\varphi\|_{p,k} \cong \|P\varphi\|_{p,k} \quad \text{for all } \varphi \in C_\pi^\infty.$$

In view of (4.3), (4.6) and (4.7) we obtain for a large enough, say $a \cong R$ ($\cong 2c$)

$$(4.8) \quad \begin{aligned} \|(L_{p,k} + aI)\varphi - (P_{p,k} + aI)\varphi\|_{p,k} &= \|(L_{p,k} - P_{p,k})\varphi\|_{p,k} \cong \\ &\cong (\alpha + \beta/(a-c))\|(P_{p,k} + aI)\varphi\|_{p,k} \cong \alpha'\|(P_{p,k} + aI)\varphi\|_{p,k} \quad \text{for all } \varphi \in C_\psi^\infty, \end{aligned}$$

where $\alpha' \in]0, 1[$.

The inequality (4.8) implies that $D(L_{p,k}^\sim) = D(P_{p,k}^\sim)$ and that for $a \cong R$

$$(4.9) \quad \|(L_{p,k}^\sim + aI)u - (P_{p,k}^\sim + aI)u\|_{p,k} \cong \alpha'\|(P_{p,k}^\sim + aI)u\|_{p,k}$$

for all $u \in D(P_{p,k}^\sim)$. Since $P_{p,k}^\sim + aI$ is a Fredholm operator with $\text{ind}(P_{p,k}^\sim + aI) = 0$ and since by (4.9) the operator $L_{p,k}^\sim + aI$ is $(P_{p,k}^\sim + aI)$ -bounded with relative bound α' smaller than one, one sees that $L_{p,k}^\sim + aI$ is a Fredholm operator with

$$(4.10) \quad \text{ind}(L_{p,k}^\sim + aI) = 0$$

(cf. [6], p. 236). Due to (4.9) and (4.6) the range $R(L_{p,k}^\sim + aI)$ is closed and the kernel $N(L_{p,k}^\sim + aI) = \{0\}$. Thus by (4.10) $R(L_{p,k}^\sim + aI) = \mathcal{B}_{p,k}^\pi$. This finishes the proof. $\square$

For $p \in]1, \infty[$ we obtain

Corollary 4.2. Suppose that $P(\cdot)$ and $Q(\cdot): \mathbb{Z}^n \rightarrow \mathbb{C}$ obey the estimates

$$(4.11) \quad |P(l)| \cong Ck_M(l) \quad \text{and} \quad |Q(l)| \cong Ck_M(l)$$

and

$$(4.12) \quad \text{Re } P(l) \cong -c \quad \text{and} \quad \text{Re } Q(l) \cong -c \quad \text{for all } l \in \mathbb{Z}^n,$$

where C, M and c are positive constants. Furthermore, suppose that $L \in L^{1n}$ such that

$$(4.3) \quad \|(L - P)\varphi\|_{p,k} \cong \alpha\|P\varphi\|_{p,k} + \beta\|\varphi\|_{p,k}$$

and

$$(4.13) \quad \|(L' - Q)\varphi\|_{p',1/k} \cong \alpha\|Q\varphi\|_{p',1/k} + \beta\|\varphi\|_{p',1/k}$$

for all $\varphi \in C_\pi^\infty$, where $\alpha \in]0, 1[$, $\beta \cong 0$ and $Q = Q(D)$.

Then the equation

$$(4.14) \quad L_{p,k}^\sim = L_{p,k}^{\prime\sharp}$$

is valid.

PROOF. In virtue of Theorem 4.1 one has

$$(4.15) \quad R(L_{p,k}^\sim + aI) = \mathcal{B}_{p,k}^\pi \quad \text{and} \quad R(L_{p',1/k}^{\prime\sharp} + aI) = \mathcal{B}_{p',1/k}^\pi$$

for a large enough, say $a \cong R$. We show that

$$(4.16) \quad N(L_{p,k}^{\prime\sharp} + aI) = \{0\}.$$

Let u be in $N(L_{p,k}^{\prime\sharp} + aI)$ and let $U = \mathcal{L}(u)$. Then one has

$$U((L_{p',1/k}^{\prime\sharp} + aI)\varphi) = u((L' + aI)\varphi) = 0 \quad \text{for all } \varphi \in C_\pi^\infty$$

and thus

$$U((L'_{p',1/k} + aI)v) = 0 \quad \text{for all } v \in D(L'_{p',1/k}).$$

This shows that $U \in (R(L'_{p',1/k} + aI))^\perp = \{0\}$. Hence (4.16) holds. Let u be in $D(L'_{p,k})$ and let $L'_{p,k}u = f$. Choose $w \in D(L'_{p,k})$ with $(L'_{p,k} + aI)w = f + au = (L'_{p,k} + aI)u$. Since $(L'_{p,k} + aI)w = (L'_{p,k} + aI)u$ one sees by (4.16) that $u = w \in D(L'_{p,k})$ and that $L'_{p,k}u = f$. This shows (4.14). $\square$

5. The semi-bounded operator

5.1. We assume further that $L = L(x, D)$ lies in L^π . In addition, we in this chapter assume that p lies in the interval $]1, \infty[$. Then the spaces $\mathcal{B}_{p,k}^\pi$ are reflexive and so one has $L_{p,k} = L_{p,k}^{**}$ (cf. [6], p. 168).

Let k and k^* be in k_π . We show

Theorem 5.1. Suppose that $L \in L^\pi$ such that with some constants $C > 0$, $a \in \mathbb{C}$ and $c > 0$ the estimates

$$(5.1) \quad \|(L + aI)\varphi\|_{p,k} \leq C \|\varphi\|_{p,kk^*}$$

$$(5.2) \quad \|(L + aI)\varphi\|_{p,k} \geq c \|\varphi\|_{p,k}$$

and

$$(5.3) \quad \|(L' + aI)\varphi\|_{p',1/(kk^*)} \geq c \|\varphi\|_{p',1/(kk^*)} \quad \text{for } \varphi \in C_\pi^\infty$$

hold.

Then the relations

$$(5.4) \quad N(L_{p,k} + aI) = \{0\} \quad \text{and} \quad R(L_{p,k} + aI) = \mathcal{B}_{p,k}^\pi$$

are valid.

PROOF. A) In virtue of (5.2) and (5.3) one has

$$(5.5) \quad \|(L_{p,k} + aI)u\|_{p,k} \geq c \|u\|_{p,k} \quad \text{for } u \in D(L_{p,k})$$

and

$$(5.6) \quad \|(L'_{p',1/(kk^*)} + aI)v\|_{p',1/(kk^*)} \geq c \|v\|_{p',1/(kk^*)} \quad \text{for } v \in D(L'_{p',1/(kk^*)}).$$

Thus one sees that $R(L_{p,k} + aI)$ (and $R(L'_{p',1/(kk^*)} + aI)$) is closed in $\mathcal{B}_{p,k}^\pi$ (and in $\mathcal{B}_{p',1/(kk^*)}^\pi$, resp.). In addition, the relations

$$(5.7) \quad N(L_{p,k} + aI) = \{0\} \quad \text{and} \quad N(L'_{p',1/(kk^*)} + aI) = \{0\}$$

hold. Hence we get

$$(5.8) \quad \begin{aligned} R(L'_{p',1/(kk^*)} + aI) &= R(L'_{p',1/(kk^*)} + aI) = \\ &= N(L'_{p',1/(kk^*)} + aI)^\perp = \{0\}^\perp = \mathcal{B}_{p',1/(kk^*)}^{\pi*}. \end{aligned}$$

B) We show that

$$(5.9) \quad R(L_{p,kk^*} + aI) = \mathcal{B}_{p,kk^*}^\pi.$$

Let $f \in \mathcal{B}_{p,kk^*}^\pi$ and let $F = \mathcal{L}(f) \in \mathcal{B}_{p',1/(kk^*)}^{\pi*}$. Due to (5.8) we find an element U

from $\mathcal{B}_{p',1/(kk^-)}^{\pi*}$ such that

$$(5.10) \quad (L_{p',1/(kk^-)}'^* + aI)U = F$$

and then one has with $u := \mathcal{L}^{-1}(U)$

$$(5.11) \quad \begin{aligned} u((L' + aI)\varphi) &= U((L_{p',1/(kk^-)}'^* + aI)\varphi) \\ &= F(\varphi) = f(\varphi) \quad \text{for all } \varphi \in C_\pi^\infty. \end{aligned}$$

This shows that $f = (L_{p,kk^-}^\# + aI)u \in R(L_{p,kk^-}^\# + aI)$, as required.

C) It remains to show that

$$(5.12) \quad R(L_{p,k}^\sim + aI) = \mathcal{B}_{p,k}^\pi.$$

Let ψ be in C_π^∞ . Due to (5.9) we find an element $w \in \mathcal{B}_{p,kk^-}^\pi$ such that $(L_{p,kk^-}^\# + aI)w = \psi$. From (5.1) and (5.2) one sees that

$$(5.13) \quad \mathcal{B}_{p,kk^-}^\pi \subset D(L_{p,k}^\sim)$$

and then $w \in D(L_{p,k}^\sim)$ and $(L_{p,k}^\sim + aI)w = \psi$. Hence the inclusion

$$(5.14) \quad C_\pi^\infty \subset R(L_{p,k}^\sim + aI)$$

is valid. Since C_π^∞ is dense in $\mathcal{B}_{p,k}^\pi$ and since $R(L_{p,k}^\sim + aI)$ is closed, the relation (5.12) holds. This completes the proof. $\square$

Let $k, k_1^\sim$ and $k_2^\sim$ be in $\mathcal{K}_\pi$. The following corollary is obtained

Corollary 5.2. Suppose that $L \in L^{\pi\pi}$ such that with some constants $C > 0$, $a \in \mathbb{C}$ and $c > 0$ the estimates

$$(5.1) \quad \|(L + aI)\varphi\|_{p,k} \leq C \|\varphi\|_{p,kk_1^\sim}$$

$$(5.2) \quad \|(L + aI)\varphi\|_{p,k} \leq c \|\varphi\|_{p,k}$$

$$(5.3) \quad \|(L' + aI)\varphi\|_{p',1/(kk_1^\sim)} \leq c \|\varphi\|_{p',1/(kk_1^\sim)}$$

$$(5.15) \quad \|(L' + aI)\varphi\|_{p',1/k^\sim} \leq C \|\varphi\|_{p',k_2^\sim/k^\sim}$$

$$(5.16) \quad \|(L' + aI)\varphi\|_{p',1/k^\sim} \leq c \|\varphi\|_{p',1/k^\sim}$$

and

$$(5.17) \quad \|(L + aI)\varphi\|_{p',k/(k_2^\sim)} \leq c \|\varphi\|_{p',k/(k_2^\sim)} \quad \text{for all } \varphi \in C_\pi^\infty$$

hold.

Then the equality

$$(5.18) \quad L_{p,k}^\sim = L_{p,k}^{\prime\sharp}$$

is valid.

PROOF. In virtue of Theorem 5.1 one has $R(L_{p,k}^\sim + aI) = \mathcal{B}_{p,k}^\pi$ and $R(L_{p',1/k^\sim}^{\prime\sharp} + aI) = \mathcal{B}_{p',1/k^\sim}^\pi$. Thus the assertion follows as in the proof of Corollary 4.2. $\square$

5.2. In the sequel we assume that $p=2$. As we mentioned above, the spaces $\mathcal{H}_k^\pi = \mathcal{B}_{2,k}^\pi$ are Hilbert spaces with the inner product

$$(u, v)_k := (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} u_l \bar{v}_l k^2(l).$$

In the case when $k \equiv 1$ we denote $(u, v)_k = (u, v)_0$.

Theorem 5.3. Suppose that $L \in L^{1n}$ such that with some constants $C > 0$ and $c > 0$ the estimates

$$(5.1) \quad \|L\varphi\|_k \leq C \|\varphi\|_{kk^\sim}$$

$$(5.19) \quad \operatorname{Re}((L + cI)\varphi, k^2(D)\varphi)_0 \geq 0$$

and

$$(5.20) \quad \operatorname{Re}((L + cI)\varphi, (kk^\sim)^2(D)\varphi)_0 \geq 0 \quad \text{for all } \varphi \in C_\pi^\infty,$$

where $k^\sim \geq 1$.

Then the relations

$$(5.4) \quad N(L_{2,k}^\sim + aI) = \{0\} \quad \text{and} \quad R(L_{2,k}^\sim + aI) = \mathcal{H}_k^\pi$$

are valid for $a > c$.

PROOF. In virtue of (5.19) one has for $a > c$

$$(5.21) \quad \begin{aligned} \operatorname{Re}((L + aI)\varphi, \varphi)_k &= \operatorname{Re}((L + aI)\varphi, k^2(D)\varphi)_0 \geq \\ &\geq -c(\varphi, k^2(D)\varphi)_0 + a(\varphi, k^2(D)\varphi)_0 = (a - c)\|\varphi\|_k^2 \end{aligned}$$

and then

$$(5.22) \quad \| (L + aI)\varphi \|_k \geq (a - c)\|\varphi\|_k \quad \text{for all } a > c.$$

Similarly we obtain

$$(5.23) \quad \operatorname{Re}((L + aI)\varphi, (kk^\sim)^2(D)\varphi)_0 \geq (a - c)\|(kk^\sim)(D)\varphi\|_0^2$$

and then

$$(5.24) \quad \operatorname{Re}((L + aI)(kk^\sim)^{-2}(D)\varphi, \varphi)_0 \geq (a - c)\|(kk^\sim)^{-1}(D)\varphi\|_0^2 = (a - c)\|\varphi\|_{1/(kk^\sim)}^2.$$

One has

$$(5.25) \quad \begin{aligned} |\operatorname{Re}((L + aI)(kk^\sim)^{-2}(D)\varphi, \varphi)_0| &= |\operatorname{Re}((L + aI)(kk^\sim)^{-2}(D)\varphi)(\bar{\varphi})| = \\ &= |\operatorname{Re}((kk^\sim)^{-2}(D)\varphi)((L' + aI)\bar{\varphi})| = \\ &= |\operatorname{Re}((kk^\sim)^{-2}(D)\varphi, \overline{(L' + aI)\bar{\varphi}})_0| \leq \\ &\leq \|\varphi\|_{1/kk^\sim} \|(kk^\sim)^{-1}(D)\overline{(L' + aI)\bar{\varphi}}\|_0 = \\ &= \|\varphi\|_{1/kk^\sim} \|(L' + aI)\varphi\|_{1/(kk^\sim)^\sim}, \end{aligned}$$

where we used the fact that

$$(5.26) \quad \begin{aligned} \|\bar{\varphi}\|_k^2 &= (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} |\bar{\varphi}_l k(l)|^2 = (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} |\bar{\varphi}_{-l} k(l)|^2 = \\ &= (2\pi)^{-n} \sum_{l \in \mathbb{Z}^n} |\varphi_l k(-l)|^2 = \|\varphi\|_{k^\sim}^2. \end{aligned}$$

Hence by (5.24) we obtain

$$(5.27) \quad \|(L' + aI)\varphi\|_{1/(kk^*)} \cong (a - c)\|\bar{\varphi}\|_{1/kk^*} = (a - c)\|\varphi\|_{1/(kk^*)}$$

for $a > c$.

Finally we note by (5.1) that

$$(5.28) \quad \|(L + aI)\varphi\|_k \cong (C + a)\|\varphi\|_{kk^*}$$

since $k^* \cong 1$. Hence all of the assumptions of Theorem 5.1 hold. This completes the proof. $\square$

Corollary 5.4. Suppose that $L \in L'^\pi$ such that with some constants $C > 0$ and $c > 0$ the estimates

$$(5.1) \quad \|L\varphi\|_k \cong C\|\varphi\|_{kk^*}$$

$$(5.29) \quad \|L\varphi\|_{k/k^*} \cong C\|\varphi\|_k$$

$$(5.19) \quad \operatorname{Re}((L + cI)\varphi, k^2(D)\varphi)_0 \cong 0$$

$$(5.20) \quad \operatorname{Re}((L + cI)\varphi, (kk^*)^2(D)\varphi)_0 \cong 0$$

and

$$(5.30) \quad \operatorname{Re}((L + cI)(k^*)^2(D)\varphi, k^2(D)\varphi)_0 \cong 0 \text{ for all } \varphi \in C_\pi^\infty$$

hold, where $k^* \cong 1$.

Then the equality

$$(5.31) \quad L_{2,k}^* = L_{2,k}'$$

is valid.

PROOF. In virtue of (5.1) we obtain

$$|(L'\varphi)(\psi)| = |\varphi(L\psi)| \cong \|\varphi\|_{(k^*/k)^*} \|L\psi\|_{k/k^*} \cong C\|\varphi\|_{(k^*/k)^*} \|\psi\|_k$$

and then

$$(5.32) \quad \|L'\varphi\|_{1/k^*} \cong C\|\varphi\|_{k^*/k^*} \text{ for all } \varphi \in C_\pi^\infty.$$

Furthermore, one sees by (5.19) that

$$\operatorname{Re}((L + cI)(k^{-2}(D)\psi), \psi)_0 \cong 0 \text{ for all } \psi \in C_\pi^\infty$$

(choose $\varphi = k^{-2}(D)\psi \in C_\pi^\infty$ in (5.19)). Hence one has

$$(5.33) \quad \begin{aligned} \operatorname{Re}((L' + cI)\psi, (k)^{-2}(D)\psi)_0 &= \\ &= \operatorname{Re}((L' + cI)\psi, \overline{(k^*)^{-2}(D)\psi}) = \operatorname{Re}((L' + cI)\psi, (k^{-2}(D)\bar{\psi})) = \\ &= \operatorname{Re}\psi((L + cI)k^{-2}(D)\bar{\psi}) = \operatorname{Re}((L + cI)(k^{-2}(D)\bar{\psi}), \bar{\psi})_0 \cong 0 \end{aligned}$$

for $\psi \in C_\pi^\infty$. Similarly one sees by (5.30) that (at first choose $\varphi = k^{-2}(D)\psi$ and then evaluate as in (5.33))

$$(5.34) \quad \operatorname{Re}((L' + cI)\psi, ((k^*/k)^*)^2(D)\psi)_0 \cong 0 \text{ for } \psi \in C_\pi^\infty.$$

The inequalities (5.1), (5.19) and (5.20) imply by Theorem 5.3 that $R(L_{2,k}^* + aI) = \mathcal{H}_k^\pi$ for $a > c$. Similarly the inequalities (5.32), (5.33) and (5.34) imply by Theorem

5.3 that $R(L'_{2,1/k^\sim} + aI) = \mathcal{H}_{1/k^\sim}^\pi$ for $a > c$. Hence the identity (5.31) is showed as in Corollary 4.2. $\square$

The Corollary 5.4 can be also formulated as follows (cf. the computation performed in (5.33))

Corollary 5.5. *Suppose that $L \in L^{t\pi}$ such that with some constants $C > 0$ and $c > 0$ the estimates*

$$(5.1) \quad \|L\varphi\|_k \leq C \|\varphi\|_{kk^\sim}$$

$$(5.29) \quad \|L\varphi\|_{k/k^\sim} \leq C \|\varphi\|_k$$

$$(5.35) \quad \operatorname{Re}((L' + cI)(k^\sim)^2(D)\varphi, \varphi)_0 \geq 0$$

$$(5.36) \quad \operatorname{Re}((L' + cI)((kk^\sim)^\sim)^2(D)\varphi, \varphi)_0 \geq 0$$

and

$$(5.30) \quad \operatorname{Re}(k^2(D)(L + cI)(k^\sim)^2(D)\varphi, \varphi)_0 \geq 0 \quad \text{for } \varphi \in C_\pi^\infty$$

hold, where $k^\sim \geq 1$.

Then the equality

$$(5.31) \quad L^\sim := L_{2,k}^\sim = L_{2,k}^{\prime\sharp} =: L_k^{\prime\sharp}$$

is valid.

The assumptions (5.35), (5.36) and (5.30) mean a semiboundedness (from below) condition for $(L' + cI)(k^\sim)^2(D)$, $(L' + cI)((kk^\sim)^\sim)^2(D)$ and $k^2(D)(L + cI)(k^\sim)^2(D)$. The assumptions (5.1) and (5.29) mean a boundedness condition for L .

The formulation of Theorem 5.3 can be similarly modified (that is, (5.19) and (5.20) can be replaced (5.35) and (5.36)).

Corollary 5.6. *Suppose that $L \in L^{t\pi}$ such that with some constants $C > 0$ and $c > 0$ the estimates*

$$(5.37) \quad \|L\varphi\|_0 \leq C \|\varphi\|_k$$

$$(5.35) \quad \|L\varphi\|_{1/k^\sim} \leq C \|\varphi\|_0$$

$$(5.39) \quad \operatorname{Re}((L + cI)\varphi, \varphi)_0 \geq 0$$

$$(5.40) \quad \operatorname{Re}((L' + cI)((k^\sim)^\sim)^2(D)\varphi, \varphi)_0 \geq 0$$

and

$$(5.41) \quad \operatorname{Re}((L + cI)(k^\sim)^2(D)\varphi, \varphi)_0 \geq 0$$

are valid, where $k^\sim \geq 1$.

Then the equality

$$(5.42) \quad L^\sim := L_{2,1}^\sim = L_{2,1}^{\prime\sharp} =: L^{\prime\sharp}$$

holds.

Remark 5.7. The inequalities (5.35), (5.37) and (5.38) imply that $N(L^\sim + aI) = \{0\}$ and that $R(L^\sim + aI) = \mathcal{H}^\pi$ for $a > c$. $\mathcal{H}^\pi$ is the space of all periodic $L^2(W)$ -(Lebesgue) integrable functions.

5.3. Suppose that L is an elliptic partial differential operator with C_π^∞ -coefficients, that is,

$$L(x, D) = \sum_{|\sigma| \leq 2m} a_\sigma D^\sigma, \quad m \in \mathbf{N}, a_\sigma \in \mathbf{C}.$$

Then for each $t \in \mathbf{N}_0$ there exists a constant $c > 0$ and $\gamma > 0$ such that

$$(5.43) \quad \operatorname{Re}((L(x, D) + c)\varphi, \varphi)_0 \geq \gamma \|\varphi\|_{k_{2m}} \quad \text{for all } \varphi \in C_\pi^\infty$$

(cf. [3], p. 117). Let t be in $\mathbf{N}_0$. Choosing $\bar{k}_t(l) = (\sum_{|\alpha| \leq t} l^{2\alpha})^{1/2}$ and $k^\sim(l) = \sum_{|\alpha| \leq m} l^{2\alpha}$ one sees that $L'(x, D)\bar{k}_t^2(D)$, $L'(x, D)(\bar{k}_t k^\sim)^2(D)$ and $\bar{k}_t^2(D)L(x, D)(k^\sim)^2(D)$ are also elliptic operators (of order $2(m+t)$ and $2(2m+t)$, resp.). Hence by (5.43) there exists a constant $c > 0$ such that (note that $k^\sim \geq 1$)

$$\operatorname{Re}(L'(x, D)\bar{k}_t^2(D)\varphi, \varphi)_0 + c(\bar{k}_t^2(D)\varphi, \varphi)_0 \geq 0,$$

$$\operatorname{Re}(L'(x, D)(\bar{k}_t k^\sim)^2(D)\varphi, \varphi)_0 + c((\bar{k}_t k^\sim)^2(D)\varphi, \varphi)_0 \geq 0$$

and

$$\operatorname{Re}(\bar{k}_t^2(D)L(x, D)(k^\sim)^2(D)\varphi, \varphi)_0 + c((\bar{k}_t k^\sim)^2(D)\varphi, \varphi)_0 \geq 0.$$

Hence the inequalities (5.35), (5.36) and (5.30) hold. The inequalities (5.1) and (5.29) are also valid (cf. [3], p. 111).

Using the evaluation such as in (5.33)–(5.34) one sees that the inequalities (5.35), (5.36) and (5.30) hold also in the case when $k = (\bar{k}_t)^{-1}$. For example, (5.30) is proved as follows: The operator $\bar{k}_t^2(D)k^{\sim 2}(D)L'(x, D)$ is elliptic. Hence there exists $c \geq 0$ such that

$$\operatorname{Re}(\bar{k}_t^2(D)k^{\sim 2}(D)L'(x, D)\varphi, \varphi)_0 + c(\bar{k}_t^2(D)k^{\sim 2}(D)\varphi, \varphi)_0 \geq 0$$

and then

$$\operatorname{Re}(k^{\sim 2}(D)(L'(x, D) + c)\varphi, \bar{k}_t^2(D)\varphi)_0 \geq 0.$$

Choosing $\varphi = \bar{k}_t^{-2}(D)\psi$ one has

$$\operatorname{Re}(k^{\sim 2}(D)(L'(x, D) + c)\bar{k}_t^2(D)\psi, \psi)_0 \geq 0.$$

Thus one has (cf. (5.33) and note that $k^{\sim\sim} = k^\sim$ and $\bar{k}_t^{\sim} = \bar{k}_t$)

$$\operatorname{Re}(\bar{k}_t^{-2}(D)\bar{\psi}, (L(x, D) + c)k^{\sim 2}(D)\bar{\psi})_0 \geq 0,$$

which implies (5.30).

Since $\bar{k}_t$ and k_t are equivalent weight functions we obtain

Corollary 5.8. Let $L(x, D)$ be an elliptic partial differential operator with C_π^∞ -coefficients and let t be in $\mathbf{Z}$. Then one has

$$(5.44) \quad N(L_{k_t}^\# + aI) = \{0\} \quad \text{and} \quad R(L_{k_t}^\sim + aI) = \mathcal{H}_{k_t}^\pi$$

for a large enough and

$$(5.45) \quad L_{k_t}^\sim = L_{k_t}^{\prime\#}.$$

For elliptic operators one has

$$\|L(x, D)\varphi\|_{k_t} + c\|\varphi\|_{k_t} \geq \gamma\|\varphi\|_{k_{t+2m}} \quad \text{for } \varphi \in C_\pi^\infty.$$

Hence we obtain from (5.45)

$$D(L_{k_t}^{\#}) = D(L_{k_t}) \subset \mathcal{H}_{k_t+2m}^{\pi}.$$

We remark that in the proof of Corollary 5.7 we need only the validity of

$$\operatorname{Re}(L(x, D)\varphi, \varphi)_0 + c(\varphi, \varphi)_0 \geq 0 \quad \text{for } \varphi \in C_{\pi}^{\infty}$$

for elliptic operators; we not need the strong estimate (5.43).

6. References

- [1] R. BEALS, Advanced Mathematical Analysis. Springer-Verlag, New York—Heidelberg—Berlin, 1973.
- [2] L. BERS and M. SCHECHTER, Elliptic equations. Partial differential equations (by L. Bers, F. John and M. Schechter). Interscience Publishers, a division of John Wiley Sons, Inc., New York—London—Sydney, 1964.
- [3] J. EELS, Elliptic operators on manifolds. Complex analysis and its applications. International Centre for Theoretical Physics, Trieste, Internat. Atomic Energy Agency, Vienna, 1976, pp. 95—152.
- [4] A. FRIEDMAN, Partial differential operators. Holt Rinehart and Wiston, Inc., New York—Chicago—San Francisco—Atlanta—Dallas—Montreal—Toronto—London—Sydney, 1969.
- [5] W. GREUB, Linear Algebra. Springer-Verlag, New York—Heidelberg—Berlin, 1981.
- [6] T. KATO, Perturbation theory for linear operators. Springer-Verlag, Berlin—Heidelberg—New York, 1966.
- [7] A. A. KULESHOV, Linear equations in space of periodic generalized functions. *Differents. Uravn.*, **20**, No. 2, 1984.
- [8] I. S. LOUHIVAARA and C. G. SIMADER, Über periodische Lösungen koerziver linearer partieller Differentialgleichungen. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4**, 1979, pp. 87—108.
- [9] M. SCHECHTER, Spectral of partial differential operators. North-Holland series in applied mathematics and mechanics 14. North-Holland Publishing Company, Amsterdam—London, 1971.
- [10] J. TERVO, On one-sided invertibility of linear coercive Fourier series operators. *Publ. Math. Debrecen* **34** (1987), 43—60.
- [11] J. TERVO, On compactness of Fourier series operators. *Ber Univ. Jyväskylä Math. Inst. Ber.* **33**, 1987.
- [12] K. YOSIDA, Functional analysis. Springer-Verlag, Berlin—Heidelberg—New York, 1974.

(Received May 26, 1987)