

A note on constructing E -rings

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1. Introduction

E -rings were introduced by PHILLIP SCHUTZ in 1973 as those rings R for which every endomorphism of their additive group R^+ can be achieved by left multiplication with some fixed ring element [S]. While appearing to be a rather specialized class of rings, E -rings turned up naturally in abelian group theory. For example, given a torsion-free abelian group G of finite rank, the center $R = \text{Center}(\text{End}G)$ of its endomorphism ring $\text{End}G$ is an E -ring in each of the following cases:

- (1) G is strongly irreducible [R];
- (2) G is strongly homogeneous [A, K, H3];
- (3) G is reduced E -uniserial [H1].

In each of these cases, G is (or is very close to) a free module over an E -ring.

BOWSHELL and SCHULTZ characterized the finite-rank torsion-free E -rings as precisely those rings which are quasi-isomorphic to $R_1 \times R_2 \times \dots \times R_n$ where each R_i is a strongly indecomposable subring of an algebraic number field and $\text{Hom}_Z(R_i, R_j) = 0$ for $i \neq j$ [BS]. In this context, E -rings had arisen earlier in the investigations of BEAUMONT and PIERCE on torsion-free rings [BP1, BP2, P]. Until recently, the only known E -rings of infinite rank were the pure subrings of the ring $\hat{Z}_p$ of p -adic integers. The question of whether E -rings of arbitrarily large cardinalities exist was settled in the affirmative by DUGAS, MADER and VINSONHALER in 1987 [DMV]. In [DH], a torsion-free E -ring is constructed which is a valuation domain but is not Noetherian. These results indicate that E -rings exist in abundance and that their structure may be rather complicated.

The purpose of this note is to outline a method for obtaining all E -rings. This method is then used to construct all torsion-free valuation domains which are E -rings. We will conclude with a number of problems and an example.

Our notation will be standard. All rings considered are unital. Maps are written to the right. Note that every E -ring is commutative [S, p. 65,

Lemma 6]. Thus, an E -ring could also be defined as a ring R such that every endomorphism of R^+ is a *right* multiplication with some $r \in R$.

2. The E -ring core of a ring

For every ring R , BOWSHELL and SCHULTZ define a subring $T(R)$ of the center of R which they call the T -core of R [BS]. While $T(R)$ is always an E -ring, in general $T(R) \neq R$, even if R is an E -ring [BS, p. 202, 1.11]. The E -ring core of R defined below is an E -subring of the center of R which coincides with R if R is an E -ring.

For R a ring and $r \in R$, we let ρ_r be the right multiplication by r , i.e. $x\rho_r = xr$ for all $x \in R$. Define

$$\check{R} = \{r \in R \mid \rho_r \in \text{Center}(\text{End}_Z R^+)\}.$$

One verifies

Proposition 2.1. (1) $\check{R}$ is a subring of the center of R , and $\check{R} \simeq \text{Center}(\text{End}_Z R^+)$, as rings. (2) R is an E -ring if and only if $\check{R} = R$. (3) If R is a domain and $0 \neq r, s \in R$, then $s \in \check{R}$ if both $rs \in \check{R}$ and $r \in \check{R}$.

For any ring R , define subrings $C_\alpha(R)$ as follows: let $C_0(R) = R$. If $C_\mu(R)$ is defined for all $\mu < \alpha$, let $C_\alpha(R) = \bigcap_{\mu < \alpha} C_\mu(R)$ if α is a limit ordinal; if $\alpha = \mu + 1$, let $C_\alpha(R) = \check{C}_\mu(R)$. Then

$$R = C_0(R) \supseteq C_1(R) \supseteq \dots \supseteq C_\alpha(R) \supseteq C_{\alpha+1}(R) \supseteq \dots$$

is a descending series of subrings of R which we shall call the *descending C -chain of R* . For set theoretical reasons, there exist ordinals α such that $C_\alpha(R) = C_{\alpha+1}(R)$. The least α with $C_\alpha(R) = C_{\alpha+1}(R)$ is called the *E -ring length of R* ; the ring $C_\alpha(R) = C_{\alpha+1}(R)$ is called the *E -ring core of R* and is denoted by $C_\infty(R)$.

We record the following result.

Theorem 2.2.. For each ring R , the E -ring core $C_\infty(R)$ of R is an E -ring.

3. E -Rings which are valuation domains

As an illustration, we shall construct all torsion-free valuation domains which are E -rings. By a valuation domain we mean an integral domain with linearly ordered ideal lattice.

Theorem 3.1. *Let R be a torsion-free valuation domain. Then $C_\infty(R)$ is an E -ring which is a torsion-free valuation domain.*

PROOF. If R^+ is divisible, $\check{R} \simeq Q$ is an E -ring with linearly ordered ideal lattice. Assume $R^+ = D \oplus H$ with D divisible and $H \neq 0$ reduced. We claim that $\check{R}$ is a reduced valuation domain. Let $\pi : D \oplus H \rightarrow H$ be the projection onto H along D . If $r \in \bigcap_{n \in \mathbb{N}} n\check{R}$ and $0 \neq h \in H$,

$$hr = h\pi r = hr\pi \in \left(\bigcap_{n \in \mathbb{N}} nR \right) \pi \subseteq D\pi = 0,$$

so that $r = 0$. Hence $\check{R}$ is reduced. Let $s, t \in \check{R}$. Since R is a valuation domain, there is no loss of generality in assuming $Rs \subseteq Rt$. Hence $s = rt$ for some $r \in R$. By 2.1, $r \in \check{R}$, which implies $\check{R}s \subseteq \check{R}t$ as desired. Thus, $C_1(R)$ is a valuation domain with reduced additive group. We claim that, for all α , $C_\alpha(R)$ is a valuation domain. Assume, inductively, that λ is an ordinal less than or equal to the E -ring length of R such that, for all $\mu < \lambda$, $C_\mu(R)$ is a valuation domain. We need to distinguish cases. Suppose $\lambda = \mu + 1$. If $\mu = 0$ this has been established already. Assume $\mu \geq 1$. Then $C_\mu(R)^+$ is reduced. Note that the additive group of any ring with linearly ordered ideal lattice is E -uniserial, i.e. the lattice of fully invariant subgroups is a chain. Since $C_\lambda(R) = \check{C}_\mu(R) \simeq \text{Center}(\text{End} C_\mu(R)^+)$, it follows from [H2, 3.5] that $C_\lambda(R)$ is a valuation domain. Suppose λ is a limit ordinal. Let $0 \neq a, b \in C_\lambda(R) = \bigcap_{\mu < \lambda} C_\mu(R)$. If, for all $\mu < \lambda$, $aC_\mu(R) = bC_\mu(R)$, then $a = bu$ with $u \in C_\mu(R)$ a unit. It follows that u is a unit in $C_\lambda(R)$ which implies $aC_\lambda(R) = bC_\lambda(R)$. Suppose there is $\mu < \lambda$ such that $aC_\mu(R) \neq bC_\mu(R)$. Without loss of generality, assume $aC_\mu(R) \subset bC_\mu(R)$. Note that this implies $aC_\nu(R) \subseteq bC_\nu(R)$ for all $\nu < \mu < \lambda$. For the same reason we must have $aC_\sigma(R) \subseteq bC_\sigma(R)$ for all $\sigma < \lambda$. Since R is a domain, $aC_\lambda(R) \subseteq bC_\lambda(R)$. It follows that $C_\lambda(R)$ is a valuation domain as claimed.

4. Problems and examples

Let G be a torsion-free abelian group, and let $R = \text{Center}(\text{End} G)$ be the center of its endomorphism ring. If, given any nonzero fully invariant subgroup F of G , the quotient group G/F is bounded, G is said to be *strongly irreducible* [R]. For G strongly irreducible, R is an integral domain, $Q \otimes_Z R$ is a field, and G is a torsion-free R -module [H2, 2.2]. Moreover, R is a *strongly irreducible ring*, i.e. every nonzero ideal I of R contains a positive integer. In particular, $\check{R}$ is a strongly irreducible domain, too. Reid has shown that $\check{R}$ is an E -ring if G has finite rank [R]. We pose

Problem 4.1. Must the E -ring core of a strongly irreducible domain be strongly irreducible ?

A similar situation occurs for *strongly homogeneous* (E -transitive) groups: G is said to be *strongly homogeneous* (E -transitive) if $\text{Aut}G$ ($\text{End}G$) operates transitively on the pure rank-one subgroups of G [A, H3]. If G is E -transitive, R is a *strongly homogeneous ring*, i.e. R is a principal ideal domain with every ideal generated by a rational integer. Thus, R^+ is a torsion-free and strongly homogeneous group, too, so that $\check{R}$ is another strongly homogeneous domain. If G has finite rank, $R = \check{R}$ is an E -ring. We ask

Problem 4.2. Must the E -ring core of a strongly homogeneous integral domain be strongly homogeneous ?

Only recently, DUGAS and SHELAH have constructed strongly homogeneous and E -transitive groups with the property that the centers of their endomorphism rings are not E -rings: assuming Gödel's axiom of constructibility $V=L$, they show the existence of strongly homogeneous (and of E -transitive, but *not* strongly homogeneous) torsion-free abelian groups G with $R = \text{Center}(\text{End}G)$ any prescribed cotorsion-free strongly homogeneous integral domain [DS]. Thus, the ring R of Example 4.7 below occurs as center of such an endomorphism ring without being an E -ring. However, $\check{R} = C_1(R)$ is an E -ring.

Problem 4.3. Is $C_\infty(R) = C_1(R)$ for any strongly homogeneous integral domain R ?

More general questions are:

Problem 4.4. If R is the center of the endomorphism ring of a torsion-free abelian group, under what conditions is $C_\infty(R) = C_1(R)$?

Problem 4.5. Given any ordinal λ , does there exist a ring R with E -ring length equal to λ ?

For a torsion-free abelian group G , if G is either strongly irreducible or reduced E -uniserial or E -transitive, the additive group of the center of its endomorphism ring has the same property. This allowed us to show that the E -ring core of a torsion-free valuation domain is a valuation domain.

Problem 4.6. What ring theoretical properties of R are inherited by its E -ring core?

We conclude with an example of a valuation domain R which is a strongly homogeneous principal ideal domain and has the property that $R \neq \check{R} = C_\infty(R)$.

Example 4.7. The p -height is a valuation on the field Q of rational numbers with value group the additive group Z of integers. Extend this valuation to the polynomial ring $Q[x]$ by defining $v(\sum_{i=0}^n a_i x^i) =$

$\min\{h_p(a_i) + i \mid 0 \leq i \leq n\}$. By [G, p. 212, 18.4], v extends to a valuation on the quotient field K of $Q[x]$ with value group Z . Let $R = \{fg^{-1} \mid f, g \in Q[x], g \neq 0, v(f) \geq v(g)\}$ be the corresponding valuation ring. Since Z has a smallest strictly positive element, R is a principal ideal domain with maximal ideal pR [G, p. 193]. It follows that R is a strongly homogeneous domain. As usual, define the derivative of $f = \sum_{i=0}^n a_i x^i \in Q[x]$ by $f' = \sum_{i=1}^n i a_i x^{i-1}$, and let $(fg^{-1})\delta = \frac{f'g - fg'}{g^2}$. One verifies that $\delta \in \text{End}_Z R^+$. Let $h \in \check{R}$. Then $\rho_h \delta = \delta \rho_h$. It follows that $h\delta = 0$ and $h = a \in Z_p$ is a constant. Thus, $\check{R} = Z_p$, the ring of integers localized at p , and $\check{R} = C_1(R) = C_\infty(R)$ is an E -ring.

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