

A note on the diophantine equation $D_1x^2 + D_2 = 2y^n$

By MAOHUA LE (Zhangjiang)

Abstract. Let D_1, D_2 be positive integers such that $2 \nmid D_1D_2$, $\gcd(D_1, D_2) = 1$, $D_1D_2 \not\equiv 7 \pmod{8}$ and D_1, D_2 are square free. Let h denote the class number of $\mathbb{Q}(\sqrt{-D_1D_2})$. In this note we prove that the equation $D_1x^2 + D_2 = 2y^n$, $x, y, n \in \mathbb{N}$, $\gcd(x, y) = 1$, $n > 1$, $\gcd(n, 2h) = 1$, has only finitely many non-trivial solutions (x, y, n) . Moreover, if (x, y, n) is a non-trivial solution, then n is an odd prime with $7 \leq n < 212603$ and $y < \exp \exp 24.17$.

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let D_1, D_2 be positive integers such that $2 \nmid D_1D_2$, $\gcd(D_1, D_2) = 1$, $D_1D_2 \not\equiv 7 \pmod{8}$ and D_1, D_2 are square free. Let h denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D_1D_2})$. In [4], LJUNGGREN discussed the solvability of the equation

$$(1) \quad \begin{aligned} D_1x^2 + D_2 &= 2y^2, \quad x, y, n \in \mathbb{N}, \quad y > 1, \quad \gcd(x, y) = 1, \\ n &> 1, \quad \gcd(n, 2h) = 1. \end{aligned}$$

In this note, using Baker's method, we prove a general result concerning (1).

Mathematics Subject Classification: 11D61, 11J86.

Key words and phrases: exponential diophantine equation, non-trivial solution, Baker's method.

Supported by the National Natural Science Foundation of China and the Guangdong Provincial Natural Science Foundation.

For any non-negative integer m , let F_m, L_m denote the m th Fibonacci number and the m th Lucas number respectively. Before proceeding we note that if

$$(2) \quad D_1 r^2 = \frac{1}{2} \left(s + (-1)^{(s-1)/2} \right), \quad D_2 = \frac{1}{2} \left(3s - (-1)^{(s-1)/2} \right), \\ 3 \nmid h, \quad r, s \in \mathbb{N}, \quad s > 1, \quad 2 \nmid s,$$

or

$$(3) \quad D_1 r^2 = \begin{cases} \frac{1}{2} F_{6k-3}, \\ \frac{1}{2} F_{6k+3}, \end{cases} \quad D_2 = \frac{1}{2} L_{6k}, \quad 5 \nmid h, \quad k, r \in \mathbb{N},$$

then (1) has solutions

$$(4) \quad (x, y, n) = \left(\frac{r}{2} (3D_2 - D_1 r^2), s, 3 \right),$$

or

$$(5) \quad (x, y, n) = \\ = \begin{cases} \left(\frac{r}{4} (D_1^2 r^4 - 10D_1 D_2 r^2 + 5D_2^2), F_{6k-1}, 5 \right), & \text{if } D_1 r^2 = \frac{1}{2} F_{6k-3}, \\ \left(\frac{r}{4} (-D_1^2 r^4 + 10D_1 D_2 r^2 - 5D_2^2), F_{6k+1}, 5 \right), & \text{if } D_1 r^2 = \frac{1}{2} F_{6k+3}. \end{cases}$$

The solutions (4) and (5) are called the trivial solutions of (1). This implies that (1) has possible infinitely many trivial solutions. For the non-trivial solutions of (1), we prove the following result.

Theorem. *The equation (1) has only finitely many non-trivial solutions (x, y, n) . Moreover, if (x, y, n) is a non-trivial solution of (1), then n is an odd prime with $7 \leq n < 212603$ and $y < \exp \exp \exp 24.17$.*

When $D_1 = 7$ and $D_2 = 11$, (1) has a non-trivial solution $(x, y, n) = (1169, 9, 7)$. This is the only example of non-trivial solutions that we know. It is natural to conjecture that $(x, y, n) = (1169, 9, 7)$ is the only non-trivial solution of (1).

2. Preliminaries

Lemma 1 ([3, Formula 3.76]). *For any positive integer n and any complex numbers α and β , we have*

$$\alpha^n + \beta^n = \sum_{i=0}^{[n/2]} (-1)^i \begin{bmatrix} n \\ i \end{bmatrix} (\alpha + \beta)^{n-2i} (\alpha\beta)^i,$$

where

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{(n-i-1)!n}{(n-2i)!i!}, \quad i = 0, \dots, [n/2]$$

are positive integers.

Let α be an algebraic number with the minimal polynomial

$$a_0z^d + a_1z^{d-1} + \dots + a_d = a_0 \prod_{i=1}^d (z - \sigma_i\alpha), \quad a_0 > 0,$$

where $\sigma_1\alpha, \dots, \sigma_d\alpha$ are conjugates of α . Then

$$h(\alpha) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log \max(1, |\sigma_i\alpha|) \right)$$

is called Weil's height of α .

Lemma 2 ([2, Theorem 3]). *Let $\varepsilon = (X_1\sqrt{D_1} + \sqrt{-D_2})/\sqrt{2}$ and $\bar{\varepsilon} = (X_1\sqrt{D_1} - \sqrt{-D_2})/\sqrt{2}$ for some positive integers X_1 . Let $\alpha = \varepsilon/\bar{\varepsilon}$ and $\Lambda = n \log \alpha - k\pi\sqrt{-1}$ for some integers n, k with $0 \leq |k| < n$. If $\Lambda \neq 0$, then we have*

$$\log |\Lambda| \geq -9AB^2,$$

where $A = \max(20, 12.85|\log \alpha| + h(\alpha))$, $d = [\mathbb{Q}(\alpha) : \mathbb{Q}]/2$,
 $B = \max(17, d \log n(1/2A + 1/25.7\pi) + 4.6d + 3.25)$.

Lemma 3 ([1, Theorem 3]). *Let $k \in \mathbb{Z}$ with $k \neq 0$, and let $F(X, Y) \in \mathbb{Z}[X, Y]$ be an irreducible binary form of degree r with $r \geq 3$. Then all solutions (X, Y) of the equation*

$$f(X, Y) = k, \quad X, Y \in \mathbb{Z}, \quad \gcd(X, Y) = 1,$$

satisfy

$$\max(|X|, |Y|) < \exp\left(3^{3(r+9)} r^{18(r+1)} H^{2r-2} (\log H)^{2r-1} \log |k|\right),$$

where H is the maximum absolute value of the coefficients of f .

3. Proof of Theorem

Let (x, y, n) be a solution of (1). Since $D_1 D_2 \not\equiv 7 \pmod{8}$ and $\gcd(n, 2h) = 1$, it follows from the analysis in [3] that $2 \nmid y$ and that there exist suitable positive integers X_1 and Y_1 such that

$$(6) \quad D_1 X_1^2 + D_2 Y_1^2 = 2y, \quad \gcd(X_1, Y_1) = 1, \quad 2 \nmid X_1 Y_1,$$

$$(7) \quad x\sqrt{D_1} + \sqrt{-D_2} = \frac{\lambda_1}{2^{(n-1)/2}} \left(X_1 \sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2} \right)^n, \\ \lambda_1, \lambda_2 \in \{-1, 1\}.$$

Let $\alpha = \lambda_1(X_1 \sqrt{D_1} + \lambda_2 Y_1 \sqrt{-D_2})/\sqrt{2}$ and $\beta = \lambda_1(X_1 \sqrt{D_1} - \lambda_2 Y_1 \sqrt{-D_2})/\sqrt{2}$. Since $\alpha - \beta = \lambda_1 \lambda_2 Y_1 \sqrt{-2D_2}$ and $\alpha\beta = y$, by (7), we get

$$(8) \quad \sqrt{-2D_2} = \alpha^n - \beta^n = \lambda_1 \lambda_2 Y_1 \sqrt{-2D_2} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right).$$

Applying Lemma 1, we obtain from (8) that

$$1 = \lambda_1 \lambda_2 Y_1 \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (\alpha - \beta)^{n-2i-1} (\alpha\beta)^i \\ = \lambda_1 \lambda_2 Y_1 \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (-2D_2 Y_1^2)^{(n-1)/2-i} y^i.$$

It follows that $Y_1 = 1$ and

$$(9) \quad \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} (-2D_2)^{(n-1)/2-i} y^i = \pm 1.$$

Let $\varepsilon = (X_1\sqrt{D_1} + \sqrt{-D_2})/\sqrt{2}$ and $\bar{\varepsilon} = (X_1\sqrt{D_1} - \sqrt{-D_2})/\sqrt{2}$. For any $m \in \mathbb{N}$ with $2 \nmid m$, let $Y_m = (\varepsilon^m - \bar{\varepsilon}^m)/(\varepsilon - \bar{\varepsilon})$. From (9), we get

$$(10) \quad \frac{\varepsilon^n - \bar{\varepsilon}^n}{\varepsilon - \bar{\varepsilon}} = \pm 1.$$

If n is not a prime, then n has an odd prime factor p and $n = pq$, where $q \in \mathbb{N}$ with $q > 1$ and $2 \nmid q$. By Lemma 1, we get from (10) that

$$Y_p \left(\frac{(\varepsilon^p)^q - (\bar{\varepsilon}^p)^q}{\varepsilon^p - \bar{\varepsilon}^p} \right) = Y_p \sum_{j=0}^{(q-1)/2} \begin{bmatrix} q \\ j \end{bmatrix} (-2D_2 Y_p^2)^{(q-1)/2-j} y^{pj} = \pm 1.$$

This implies that $Y_p = \pm 1$ and

$$(11) \quad \frac{\varepsilon^p - \bar{\varepsilon}^p}{\varepsilon - \bar{\varepsilon}} = \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-2D_2)^{(p-1)/2-i} y^i = \pm 1.$$

Similarly, by (10), we have

$$Y_q \left(\frac{(\varepsilon^q)^p - (\bar{\varepsilon}^q)^p}{\varepsilon^q - \bar{\varepsilon}^q} \right) = Y_q \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-2D_2 Y_q^2)^{(p-1)/2-i} y^{qi} = \pm 1,$$

whence we get $Y_q = \pm 1$ and

$$(12) \quad \sum_{i=0}^{(p-1)/2} \begin{bmatrix} p \\ i \end{bmatrix} (-2D_2)^{(p-1)/2-i} y^{qi} = \pm 1.$$

By (12), there exists a suitable $\lambda \in \{-1, 1\}$ such that

$$(13) \quad (-2D_2)^{(p-1)/2} - \lambda \equiv 0 \pmod{y^q}.$$

Since $\gcd((-2D_2)^{(p-1)/2} - \lambda, (-2D_2)^{(p-1)/2} + \lambda) = 1$, we see from (11) that

$$(14) \quad \begin{aligned} & \left((-2D_2)^{(p-1)/2} - \lambda\right) + p(-2D_2)^{(p-3)/2}y \\ & + p\left(\frac{p-3}{2}\right)(-2D_2)^{(p-5)/2}y^2 \equiv 0 \pmod{y^3}. \end{aligned}$$

Since $q \geq 3$ and $\gcd(y, 2D_2) = 1$, we find from (13) and (14) that $p(-2D_2 + y(p-3)/2) \equiv 0 \pmod{y^2}$, and hence, $y \mid 2D_2$, a contradiction. Thus n must be an odd prime.

If $n = 3$, then from (9) we get $-2D_2 + 3y = \pm 1$. Since $2 \nmid D_1 D_2 y$ and $y \equiv (-1)^{(y-1)/2} \pmod{4}$, we see from (6) that D_1, D_2 satisfy (2) and (1) has solutions (4).

If $n = 5$, then from (9) we get $4D_2^2 - 10D_2y + 5y^2 = \pm 1$. Then we have $(4D_2 - 5y)^2 - 5y^2 = \pm 4$, and hence,

$$(15) \quad |4D_2 - 5y| = L_m, \quad y = F_m, \quad m \in \mathbb{N}, \quad m > 1.$$

Notice that $2 \nmid L_m F_m$, $2 \parallel F_m$ and $2 \parallel L_m$ if and only if $3 \nmid m$, $m \equiv 3 \pmod{6}$ and $6 \mid m$, respectively. We find from (6) and (15) that D_1, D_2 satisfy (3) and (1) has solutions (5). Therefore, if (x, y, n) is a non-trivial solution of (1), then $n \geq 7$.

For any complex number z , we have either $|e^z - 1| > 1/2$ or $|e^z - 1| \geq |z - k\pi\sqrt{-1}|/2$ for some $k \in \mathbb{Z}$. Hence, by (10), we get

$$(16) \quad \begin{aligned} \log |\varepsilon - \bar{\varepsilon}| &= \log |\varepsilon^n - \bar{\varepsilon}^n| = n \log |\bar{\varepsilon}| + \log |(\varepsilon/\bar{\varepsilon})^n - 1| \\ &\geq n \log |\bar{\varepsilon}| + \log |n \log \alpha - k\pi\sqrt{-1}| - \log 2, \end{aligned}$$

where $k \in \mathbb{Z}$ with $|k| < n$. Let $\Lambda = n \log \alpha - k\pi\sqrt{-1}$. Since $y \geq 3$ and α satisfies

$$(17) \quad y\alpha^2 - (D_1 X_1^2 - D_2)\alpha + y = 0,$$

α is not a root of unity, and hence, $\Lambda \neq 0$. Notice that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$, $0 < |\log \alpha| < \pi$ and $h(\alpha) = \log \sqrt{y}$ by (17). Applying Lemma 2, we get

$$(18) \quad \begin{aligned} \log |\Lambda| &\geq -9(12.85\pi + \log \sqrt{y}) \\ &\times (\max(17, 7.85 + \log n(1/(25.7\pi + \log \sqrt{y}) + 1/25.7\pi)))^2. \end{aligned}$$

If $7.85 + \log 2n/25.7\pi \geq 17$, then $n > 380038$. Further, by (16) and (18), we get

$$\log \sqrt{2y} + 9(12.85\pi + \log \sqrt{y})(7.85 + \log 2n/25.7\pi)^2 > n \log \sqrt{y}.$$

It follows that

$$\begin{aligned} & \log \sqrt{2} + 735.65(7.85 + \log 0.0247712n)^2 \\ & > \log \sqrt{2} + 9 \left(\frac{12.85\pi}{\log \sqrt{y}} + 1 \right) \left(7.85 + \log \frac{2n}{25.7\pi} \right)^2 > n, \end{aligned}$$

whence we conclude $n < 380000$, a contradiction. So we have $7.85 + \log 2n/25.7\pi < 17$ and

$$(19) \quad \log |\Lambda| \geq -2601(12.85\pi + \log \sqrt{y}).$$

The combination of (16) and (19) yields

$$(20) \quad n \leq 212603.$$

Let

$$f(X, Y) = \sum_{i=0}^{(n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} X^{(n-1)/2-i} Y^i.$$

Notice that n is an odd prime and

$$\begin{bmatrix} n \\ (n-1)/2 \end{bmatrix} = n, \quad n \mid \begin{bmatrix} n \\ j \end{bmatrix}, \quad j = 1, \dots, (n-1)/2.$$

Then $f(X, Y) \in \mathbb{Z}[X, Y]$ is an irreducible binary form of degree $(n-1)/2$ with

$$H = \max_{i=0, \dots, (n-1)/2} \begin{bmatrix} n \\ i \end{bmatrix} = \max_{i=0, \dots, (n-1)/2} \frac{n}{n-2i} \binom{n-i-1}{i} < 2^{n-1}.$$

We see from (9) that $(X, Y) = (-2D_2, Y)$ is a solution of the equation

$$f(X, Y) = \pm 1, \quad X, Y \in \mathbb{Z}.$$

Therefore, by Lemma 3, if $n \geq 7$, then we have

$$(21) \quad y \leq \max(2D_2, Y) < \exp \left(3^{3(n+17)/2} \left(\frac{n-1}{2} \right)^{9(n+1)} 2^{(n-1)(n-3)} ((n-1) \log 2)^{n-3} \right).$$

Substituting (20) into (21), we deduce that $y < \exp \exp \exp 24.17$. The proof is complete.

References

- [1] Y. BUGEAUD and K. GYÖRY, Bounds for the solutions of Thue-Mahler equations and form equations, *Acta Arith.* **74** (1996), 273–292.
- [2] M. LAURENT, M. MIGNOTTE and Y. NESTERENKO, Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.
- [3] R. LIDL and H. NIEDERREITER, Finite Fields, Addison-Wesley, *Reading, MA*, 1983.
- [4] W. LJUNGGREN, On the diophantine equation $Cx^2 + D = 2y^n$, *Math. Scand.* **18** (1966), 69–86.

MAOHUA LE
DEPARTMENT OF MATHEMATICS
ZHANJIANG TEACHERS COLLEGE
POSTAL CODE 524048
ZHANJIANG, GUANGDONG
P.R. CHINA

(Received February 19, 1996)