

Central automorphisms of standard wreath products

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1. Introduction

Let $W = AW_rB$ be the standard wreath product of the groups A and B , where if $|B| = 2$ then $|A| \neq 2$ or A is not the dihedral group of order $4m + 2$. It is known that the group of central automorphisms $\text{Aut}_c(W)$ of the group W is the direct product of the groups K_c and I_c where K_c is the group of all central automorphisms which leave the group B elementwise fixed and I_c is the group of all inner central automorphisms which are induced by the elements of the base group A^B . (cf. J. PANAGOPOULOS [3]). In this paper we study the groups K_c and I_c . It is proved that the group K_c contains a normal subgroup H_c and the factor group K_c/H_c is isomorphic to a subgroup of $\text{Aut}_c(A)$. (Proposition 3.1). If the group B is finite then there is an one-one correspondence of the group H_c onto the group $\text{Hom}(A, C)$, where C is a suitable subgroup of the center $Z(A)$ of A (Proposition 3.3). As far as the group I_c is concerned, it is proved that I_c is trivial if and only if $\text{Hom}(B, Z(A))$ is trivial, where A is not of exponent 2 when $|B| = 2$. If in $W = AW_rB$ the group A is of exponent 2, $A \neq C_2$ and $|B| = 2$, then $I_c = I_1$, where I_1 is the group of inner automorphisms which are induced by the elements of the base group A^B . In the case of $W = C_2W_rC_2$ we have that $\text{Aut}_c(W) = I(W)$. (cf. J. PANAGOPOULOS [3]).

2. Preliminaries

We know that an automorphism α of A induces an automorphism α^* of $W = AW_rB$ by

$$(bf)^{\alpha^*} = bf^{\alpha^*} \quad \text{for all } b \in B, f \in A^B \quad \text{where}$$

$$f^{\alpha^*}(x) = (f(x))^\alpha \quad \text{for all } x \in B.$$

The group A^* of all such automorphisms is isomorphic to $\text{Aut}(A)$. Also

if β is an automorphism of B , then β induces an automorphism β^* of $W = AW_rB$ by

$$(bf)^{\beta^*} = b^{\beta} f^{\beta^*} \quad \text{for all } b \in B, f \in A^B, \quad \text{where}$$

$f^{\beta^*}(x) = f(x^{\beta^{-1}})$ for all $x \in B$. The group B^* of all such automorphisms is isomorphic to $\text{Aut}(B)$. (cf. C. HOUGHTON [1]).

We shall see in the following propositions the implication on α^*, β^* the property of α, β being central and on α, β the property of α^*, β^* being central.

Proposition 2.1. *If $\alpha \in \text{Aut}_c(A)$ then $(bf)^{\alpha^*} = (bf)g$, where $g \in Z(A^B)$.*

PROOF. If $f \in A^B$ then $f^{\alpha^*}(x) = (f(x))^{\alpha} = f(x)z_{f(x)}$ where $z_{f(x)} \in Z(A)$. Then, for the element $g \in A^B$ with

$$g(x) = z_{f(x)} \quad \text{for all } x \in B$$

we have $f^{\alpha^*}(x) = f(x) \cdot g(x) = (fg)(x)$ for all $x \in B$. So $f^{\alpha^*} = fg$ and $(bf)^{\alpha^*} = (bf)g$, where $g \in Z(A^B)$.

Proposition 2.2. *The automorphism α^* is central if and only if $\alpha = 1$.*

PROOF. Let $a \in A$ and $f \in A^B$ with $f(x) = a$ for some $x \in B$ and $f(y) = 1$ for all $y \in B, y \neq x$. Then

$$a^{\alpha} = (f(x))^{\alpha} = f^{\alpha^*}(x) = (f \cdot z_f)(x) = f(x) \cdot z_f(x) = a \cdot z_f(x),$$

where $z_f(x) \in Z(W) = Z(D)$ and D is the diagonal of A^B .

$$1 = f(y) = (f(y))^{\alpha} = f^{\alpha^*}(y) = (f \cdot z_f)(y) = f(y) \cdot z_f(y) = z_f(y)$$

for all $y \in B, y \neq x$.

Since $z_f \in D$ we have that $z_f(x) = 1$ and so

$$a^{\alpha} = a. \quad \text{Hence } \alpha = 1.$$

Proposition 2.3. *The automorphism β^* is central if and only if $\beta = 1$.*

PROOF. From [3] we have that $\beta^* = k \cdot i$, with $k \in K_c \leq K$ and $i \in I_c \leq I_1$, where K is the group of all the automorphisms of $W = AW_rB$ which leave the group B elementwise fixed and I_1 is the group of all the inner automorphisms which are induced by the elements of the group A^B . But it is known that the group B^* intersects trivially the group $K \cdot I_1$ (cf. C. HOUGHTON [1]). So $\beta^* = 1$ and $\beta = 1$.

Now, if $\beta \in \text{Aut}_c(B), \beta \neq 1$ then we cannot state a Proposition like Proposition 2.1.

3. The group K_c

In this section we study the relation of the group K_c to the groups A and B .

If $k \in K_c$ then $k = \alpha^* h$ with $\alpha^* \in A^*$ and $h \in H$, where H is the group of all the automorphisms of W which leave the groups B and the diagonal D elementwise fixed. (cf. C. HOUGHTON [1]. Theorem 3.3). For an element $f \in D$ we have that

$$f^{\alpha^*} = f^{k h^{-1}} = (f z_f)^{h^{-1}} = f \cdot z_f, \quad z_f \in Z(W),$$

since the automorphism h leaves the diagonal D elementwise fixed. Hence, the corresponding automorphism α of A is central. So, there is map $\varphi : K_c \rightarrow \text{Aut}_c(A)$ such that if $k = \alpha^* h \in K_c$ then $k\varphi = \alpha$. The map φ is a homomorphism. In fact, for the elements $k = \alpha^* h$ and $k_1 = \alpha_1^* h_1$ of K_c we have

$$(k k_1)\varphi = (\alpha^* h \alpha_1^* h_1)\varphi = (\alpha^* \alpha_1^* h' h_1)\varphi = \alpha \alpha_1 = (k\varphi) \cdot (k_1\varphi).$$

(This is because the group K is the semidirect product of the group H by the group A^*).

Now, if $k = \alpha^* h \in \text{Ker } \varphi$ then $\alpha = 1$, so $\alpha^* = 1$ and $k = h \in K_c \cap H$. If we define $H_c = K_c \cap H$, then it is clear that $H_c \leq \text{Ker } \varphi$ so $H_c = \text{Ker } \varphi$. Hence

$$K_c/H_c \leq \text{Aut}_c(A) \quad \text{by an isomorphism.}$$

Proposition 3.1. *If $K_c = K \cap \text{Aut}_c(W)$ and $H_c = K_c \cap H$ then H_c is normal in K_c and the factor group K_c/H_c isomorphic to a subgroup of $\text{Aut}_c(A)$.*

In the following we study the group H_c , assuming that the groups A and B are finite. If $B = \{x_1, x_2, \dots, x_k\}$, then we write

$$A^B = A_{x_1} \times A_{x_2} \times \dots \times A_{x_k}, \quad \text{where for any } i = 1, 2, \dots, k \quad \text{the}$$

group A_{x_i} consists of those elements $f_{x_i} \in A^B$ with

$$\begin{aligned} f_{x_i}(x_i) &= a \in A \quad \text{and} \\ f_{x_i}(x_j) &= 1 \quad \text{for all } x_j \neq x_i. \end{aligned}$$

If $h \in H_c$, $h \neq 1$, then for any $f_{x_i} \in A_{x_i}$ we have that

$$f_{x_i}^h = f_{x_i} z_{f_{x_i}}, \quad \text{where } z_{f_{x_i}} \in Z(W).$$

For any $x_j \neq x_i$ there is an element $b_{ij} \in B$ such that $x_j = x_i b_{ij}$. Hence,

$$\begin{aligned} f_{x_j}^h &= f_{x_i b_{ij}}^h = f_{x_i}^{b_{ij} h} = f_{x_i}^{h b_{ij}} = (f_{x_i} z_{f_{x_i}})^{b_{ij}} = f_{x_i}^{b_{ij}} z_{f_{x_i}} = \\ &= f_{x_i b_{ij}} z_{f_{x_i}} = f_{x_j} z_{f_{x_i}} \end{aligned}$$

because it is easy to see that $f_{x_i}^{b_{ij}} = f_{x_i b_{ij}}$ for any $i \neq j$ and it is known that the automorphisms which are induced by the elements of the group B , leave elementwise fixed the diagonal D . Hence, the automorphism h multiplies the elements f_{x_i} , $i = 1, 2, \dots, k$ by the same element of the center of W . If now, $f \in D$ with $f(x) = a$ for all $x \in B$ then

$$\begin{aligned} f &= f_{x_1} f_{x_2} \dots f_{x_k}, \quad \text{where} \quad f_{x_i}(x_i) = a \quad \text{and} \\ &\quad f_{x_i}(x_j) = 1, \quad x_i \neq x_j \\ &\quad \text{for all} \quad i = 1, 2, \dots, k \end{aligned}$$

Then

$$\begin{aligned} f^h &= f_{x_1}^h f_{x_2}^h \dots f_{x_k}^h = (f_{x_1} z_f)(f_{x_2} z_f) \dots (f_{x_k} z_f) = \\ &= f \cdot z_f^k, \quad \text{where} \quad z_f \in Z(W). \quad \text{Since} \quad f^h = f \quad \text{for all} \quad f \in D, \end{aligned}$$

we have that $z_f^k = 1$.

Thus, we have proved the following proposition.

Proposition 3.2. *If $W = AW_r B$ and the groups A and B are finite with $(|Z(A)|, |B|) = 1$ then $H_c = 1$.*

It remains the case $(|Z(A)|, |B|) = d$, $d \neq 1$. In this case we consider the subgroup C of $Z(A)$ which consists of all the elements whose order divides the integer d . We shall prove the following proposition.

Proposition 3.3. *There is a one-one correspondence between the group H_c and the group $\text{Hom}(A, C)$.*

PROOF. Let $h \in H_c$ and $f_{x_i} \in A_{x_i}$ for some fixed $x_i \in B$. Then

$$f_{x_i}^h = f_{x_i} z_{f_{x_i}}, \quad \text{where} \quad z_{f_{x_i}} \in Z(W).$$

If $f_{x_i}(x_i) = a \in A$ and $z_{f_{x_i}}(x) = z_a \in Z(A)$ for all $x \in B$, then we define the map $\varphi : A \rightarrow C$ with $a\varphi = z_a$ for all $a \in A$. The map φ is a homomorphism. Indeed, if $a, a' \in A$ we take the elements f_{x_i}, f'_{x_i} so that

$$f_{x_i}(x_i) = a, \quad f'_{x_i}(x_i) = a'. \quad \text{Then}$$

$$(aa')\varphi = [f_{x_i}(x_i)f'_{x_i}(x_i)]\varphi = [(f_{x_i}f'_{x_i})(x_i)]\varphi = z_a z_{a'} = (a\varphi)(a'\varphi)$$

because $(f_{x_i}f'_{x_i})^h = f_{x_i}^h f_{x_i}'^h = f_{x_i}f'_{x_i}z_{f_{x_i}}z_{f'_{x_i}}$, where

$$z_{f_{x_i}}(x) = z_a, \quad z_{f'_{x_i}}(x) = z_{a'} \quad \text{for all } x \in B.$$

Thus, we see that there is map

$$\vartheta : H_c \rightarrow \text{Hom}(A, C) \quad \text{with} \quad h^\vartheta = \varphi \quad \text{for all } h \in H_c.$$

We shall see that the map ϑ is onto the group $\text{Hom}(A, C)$. Let $\varphi \in \text{Hom}(A, C)$, with $a\varphi = z_a \in C$ for all $a \in A$. We define the map $h : A^B \rightarrow A^B$ by the rule:

$$\text{if } f_{x_i} \in A_{x_i} \quad \text{with} \quad f_{x_i}(x_i) = a, \quad \text{then}$$

we define

$$\begin{aligned} f_{x_i}^h &= f_{x_i} z_{f_{x_i}}, \quad \text{where } z_{f_{x_i}} \in D \quad \text{with} \\ z_{f_{x_i}}(x) &= a\varphi \quad \text{for all } x \in B \quad \text{and } i = 1, 2, \dots, k. \end{aligned}$$

First, we can see that the restriction of h to each A_{x_i} is a homomorphism. Indeed, if

$$f_{x_i}, g_{x_i} \in A_{x_i}, \quad \text{then}$$

$$(f_{x_i}g_{x_i})^h = f_{x_i}g_{x_i}z_{f_{x_i}g_{x_i}}, \quad \text{where}$$

$$\begin{aligned} z_{f_{x_i}g_{x_i}}(x) &= [(f_{x_i}g_{x_i})(x_i)]\varphi = [f_{x_i}(x_i)g_{x_i}(x_i)]\varphi = [f_{x_i}(x_i)]\varphi[g_{x_i}(x_i)]\varphi = \\ &= z_{f_{x_i}}(x)z_{g_{x_i}}(x) = (z_{f_{x_i}}z_{g_{x_i}})(x) \quad \text{for all } x \in B. \quad \text{Thus,} \end{aligned}$$

$$z_{f_{x_i}g_{x_i}} = z_{f_{x_i}}z_{g_{x_i}} \quad \text{and consequently}$$

$$(f_{x_i}g_{x_i})^h = (f_{x_i}z_{f_{x_i}})(g_{x_i}z_{g_{x_i}}) = f_{x_i}^h g_{x_i}^h.$$

Now, if $f, g \in A^B$ with $f = f_{x_1}^1 f_{x_2}^2 \dots f_{x_k}^k$, $g = g_{x_1}^1 g_{x_2}^2 \dots g_{x_k}^k$ where $f_{x_i}^i, g_{x_i}^i \in A_{x_i}$, $i = 1, 2, \dots, k$, then

$$\begin{aligned} (fg)^h &= (f_{x_1}^1 g_{x_1}^1)^h \dots (f_{x_k}^k g_{x_k}^k)^h = (f_{x_1}^1)^h (g_{x_1}^1)^h \dots (f_{x_k}^k)^h (g_{x_k}^k)^h = \\ &= (f_{x_1}^1 z_{f_{x_1}^1}) (g_{x_1}^1 z_{g_{x_1}^1}) \dots (f_{x_k}^k z_{f_{x_k}^k}) (g_{x_k}^k z_{g_{x_k}^k}) = \\ &= (f_{x_1}^1 \dots f_{x_k}^1) (z_{f_{x_1}^1} \dots z_{f_{x_k}^1}) (g_{x_1}^1 \dots g_{x_k}^1) (z_{g_{x_1}^1} \dots z_{g_{x_k}^1}) = \\ &= f^h g^h. \quad \text{Hence, the map } h \text{ is an endomorphism of } A^B. \end{aligned}$$

We shall now see that the endomorphism h leaves the diagonal D elementwise fixed. If $f \in D$, then

$$\begin{aligned} f &= f_{x_1} f_{x_2} \cdots f_{x_k}, \quad \text{so} \quad f^h = f_{x_1}^h f_{x_2}^h \cdots f_{x_k}^h = \\ &= (f_{x_1} z_{f_{x_1}}) \cdots (f_{x_k} z_{f_{x_k}}) = (f_{x_1} \cdots f_{x_k}) (z_{f_{x_1}} \cdots z_{f_{x_k}}) = f \cdot z^k = f, \end{aligned}$$

because we know that $z_{f_{x_1}} = \cdots = z_{f_{x_k}} = z \in Z(W)$, with $z^d = 1$, since $z(x) \in C$ for all $x \in B$ and d divides k .

Now we shall prove that h is an epimorphism. If $f_{x_i} \in A_{x_i}$ we consider the element $f \in A^B$ with $f = f_{x_1}^1 f_{x_2}^2 \cdots f_{x_k}^k$, so that

$$\begin{aligned} f_{x_i}^i(x_i) &= f_{x_i}(x_i)[f_{x_i}^{-1}(x_i)]\varphi \quad \text{and} \\ f_{x_j}^j(x_j) &= [f_{x_i}^{-1}(x_i)]\varphi \quad \text{for all } j \neq i. \end{aligned}$$

Then, we have that

$$\begin{aligned} f &= f_{x_i} z_f, \quad \text{where } z_f \in Z(W) \quad \text{with} \\ z_f(x) &= [f_{x_i}^{-1}(x_i)]\varphi. \end{aligned}$$

Thus, $f^h = (f_{x_i} z_f)^h = f_{x_i}^h z_f = f_{x_i} z_{f_{x_i}} z_f = f_{x_i}$, because $z_f \in D$ and $(z_{f_{x_i}} z_f)(x) = z_{f_{x_i}}(x) z_f(x) = [f_{x_i}(x_i)]\varphi [f_{x_i}^{-1}(x_i)]\varphi = [(f_{x_i} f_{x_i}^{-1}(x_i))]\varphi = 1$, for all $x \in B$.

From the above we conclude that h is an epimorphism of A^B . Since the group $W = AW_r B$ is finite, h is an automorphism of A^B . In order to prove that $h \in H_c$, it remains to check whether $hb = bh$ holds for all the automorphisms b which are induced by the elements of B . (ch. C. HOUGHTON [1], 3.4). Let $f_{x_i} \in A_{x_i}$ for some $x_i \in B$. Then

$$\begin{aligned} f_{x_i}^{hb} &= (f_{x_i} z_{f_{x_i}})^b = f_{x_i}^b z_{f_{x_i}} = f_{x_i b} z_{f_{x_i}} = f_{x_i b} z_{f_{x_i b}} = \\ &= f_{x_i b}^h = f_{x_i}^{bh} \end{aligned}$$

Now, it is clear that $h^\vartheta = \varphi$, so ϑ is onto the group $\text{Hom}(A, C)$. Also, it is not difficult to see that ϑ is a one-one map of H_c onto $\text{Hom}(A, C)$, but not a homomorphism.

Let now $W = AW_r B$, where the group B contains an element of infinite order. It is known that in this case the derived subgroup of W is the base group A^B . (cf. P. NEUMANN [2]). Hence, the group K_c must be trivial and theorem 3.3 in [3] gives:

Proposition 3.4. *If $W = AW_rB$ and the group B contains an element of infinite order then*

$$\text{Aut}_c(W) = I_c \quad \text{with} \quad I_c \cong \frac{Z_2(W)}{Z(W)}.$$

4. The group I_c

It is known that if $W = AW_rB$ with A of exponent $\neq 2$, when $|B| = 2$, then

$$\text{Aut}_c(W) = K_c \times I_c, \quad \text{where} \quad I_c \cong \frac{Z_2(W)}{Z(W)}.$$

(cf. J. PANAGOPOULOS [3]).

In this section we study the case : when is the group I_c trivial, so that

$$\text{Aut}_c(W) = K_c ? \quad \text{This problem is equivalent to the}$$

following: under what conditions does the group $Z_2(W)$ coincide with the group $Z(W)$?

If we apply the procedure of C. RUIZ DE VELASCO and M. TORRES in [4] to the standard wreath product $W = AW_rB$, we conclude that the group $Z_2(W)$ is isomorphic to the direct product $Z(A) \times \text{Hom}(B, Z(A))$. Let us describe this isomorphism: if $f \in Z_2(W)$ then $f_b = [b, f] \in Z(W)$, for all $b \in B$. We define the map $\varphi_f : B \rightarrow Z(A)$ by the rule

$$\varphi_f(b) = f_b(b) \quad \text{for all} \quad b \in B.$$

The map φ_f is a homomorphism. Now, for an arbitrary element $c \in B$ we define the map $F_c : Z_2(W) \rightarrow Z(A) \times \text{Hom}(B, Z(A))$ by

$$F_c(f) = (f(c), \varphi_f) \quad \text{for all} \quad f \in Z_2(W).$$

It is proved that the map F_c is an isomorphism of $Z_2(W)$ onto $Z(A) \times \text{Hom}(B, Z(A))$. We see that if $f \in Z(W)$ then

$$F_c(f) = (f(c), \varphi_f) \quad \text{with} \quad \varphi_f(b) = f_b(b) = [b, f](b) = 1 \quad \text{for all} \quad b \in B.$$

Hence, $\varphi_f = 1$ when $f \in Z(W)$, which means that

$$[Z(W)]F_c \leq Z(A).$$

Conversely if $(z, 1) \in Z(A) \times \text{Hom}(B, Z(A))$, then

$$F_c(f) = (z, 1), \quad \text{where} \quad f(x) = z \quad \text{for all} \quad x \in B.$$

This gives that $[Z(W)]F_c = Z(A)$. Hence, we see that $Z_2(W) = Z(W)$ if and only if $\text{Hom}(B, Z(A)) = 1$.

As a matter of fact we have proved the following.

Proposition 4.1. *Let $W = AW_rB$ where A is not of exponent 2, when $|B| = 2$. Then*

$$\text{Aut}_c(W) = K_c \quad \text{if and only if} \quad \text{Hom}(B, Z(A)) = 1.$$

References

- [1] C. H. HOUGHTON, On the automorphism groups of certain wreath products, *Publ. Math.* **9** (1963), 307–313, *Debrecen*.
- [2] P. M. NEUMANN, On the structure of standard wreath products of groups, *Math. Z.* **84** (1964), 343–373.
- [3] J. PANAGOPOULOS, The group of central automorphisms of the standard wreath products, *Arch. Math.* **45** (1985), 411–417.
- [4] C. RUIZ DE VELASCO and M. TORRES, *Acta Math. Hung.* **44** (3–4) (1984), 275–278.

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