

Generating functions of some products of arithmetical functions

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1. Introduction

The Dirichlet series generating function of the product of two divisor sum functions was given by S. RAMANUJAN in a paper published in 1916 [10]:

$$(1) \quad \sum \sigma_a(n) \sigma_b(n) n^{-s} = \frac{\zeta(s) \zeta(s-a) \zeta(s-b) \zeta(s-a-b)}{\zeta(2s-a-b)},$$

which holds for $\min(R(s), R(s-a), R(s-b), R(s-a-b)) > 1$. Here, and throughout the paper, the unadorned summation sign indicates the sum as n runs over the set of positive integers, and $R(s)$ denotes the real part of s . A proof of (1) was given by B. M. WILSON [15]: it is the proof which appears in [8, pp. 197-198]. Other proofs have been given by R. VAIDYANATHASAMY [14, p. 611] and by P. BUNDSCHUH [2].

The generating functions of other products of arithmetical functions have been obtained by several authors, notably S. CHOWLA [3], M. M. CRUM [4], D. REDMOND and R. SIVARAMAKRISHNAN [11] and A. MERCIER [9]. In Section 2 we will find the generating functions of products of two functions from a wide class of functions which includes Klee's function and generalized Dedekind functions, and products in which one factor is from this class and the other is a Gegenbauer function.

A Gegenbauer function $\varrho_{a,r}$ is defined by

$\varrho_{a,r}(n)$ = the sum of the a -th powers of the divisors d of n such that
 n/d is an r -th power

(see [5, p. 298]). Thus $\varrho_{a,1} = \sigma_a$. The generating function of the product $\varrho_{a,r} \varrho_{b,r}$ is

$$\frac{\zeta(rs) \zeta(r(s-a)) \zeta(r(s-b)) \zeta(s-a-b)}{\zeta(r(2s-a-b))},$$

where the appropriate Dirichlet series converges absolutely to this function in a suitable half-plane. This is equivalent to a result proved by CRUM [4, p. 11], who considered another arithmetical function which is a Gegenbauer function in disguise. In Section 3 we will use a formula due to D. M. KOTELYANSKIĬ [7] to extend Crum's result by obtaining the generating function of $\varrho_{a,r}\varrho_{b,tr}$, where t is a positive integer.

The Dirichlet product of two arithmetical functions f and g will be denoted by $f \star g$. The function ζ_a is defined by $\zeta_a(n) = n^a$, and we set $\zeta = \zeta_0$. The arithmetical function ζ has the Riemann zeta function ζ as its generating function: there is little possibility of confusing the two uses of the Greek letter zeta.

We will make use of an identity due to R. D. von STERNECK (see [5, pp. 151 – 152]) which, for two arithmetical functions f and g , states that

$$(2) \quad (f \star \zeta)(g \star \zeta) = h \star \zeta, \text{ where}$$

$$h(n) = \sum_{[a,b]=n} f(a)g(b) \quad \text{for all } n.$$

$[a, b]$ denotes the least common multiple of a and b , and the sum in (2) is over all ordered pairs of positive integers a and b with $[a, b] = n$. The function h in (2) will be denoted by $[f, g]$. Bundschuh used (2) in his proof of (1).

2. The function $\phi_{r,\alpha,a}$

Let r and α be positive integers and let a be a complex number. The arithmetical function $\phi_{r,\alpha,a}$ is defined by

$$\phi_{r,\alpha,a}(n) = \sum_{d|n} \mu_r(d)^\alpha (n/d)^a \quad \text{for all } n,$$

where

$$\mu_r(n) = \begin{cases} \mu(n^{1/r}) & \text{if } n \text{ is an } r\text{-th power} \\ 0 & \text{otherwise} \end{cases}$$

The function μ_r is the r -th convolute of the Möbius function μ . Convolutives of arithmetical functions, which were first studied systematically by VAIDYANATHASWAMY [14], occur frequently and naturally (see [8, p. 53]). If α is odd then $\mu_r^\alpha = \mu_r$, and if α is even then μ_r^α is the r -th convolute of μ^2 . Thus, we can restrict α to the values 1 and 2.

The function $\phi_{r,\alpha,a}$ was introduced by B.C. BERNDT [1] as a convenient device for considering simultaneously several well-known functions. $\phi_{1,1,1}$ is Euler's function ϕ ; $\phi_{1,1,a}$ is the function ϕ_a , and therefore Jordan's function J_a when a is a positive integer; $\phi_{r,1,1}$ is the function Φ_r , generally referred to as Klee's function because V. L. Klee gave a summary of its properties [6], but actually studied many years earlier (see [5, p. 134]). $\phi_{1,2,1}$ is Dedekind's function ψ , and $\phi_{1,2,a}$ and $\phi_{r,2,1}$ are the generalizations ψ_a and Ψ_r of ψ introduced by D. SURYANARAYANA [12], [13]. Definitions of all of these functions are given in [8] (see the index, p. 362).

Theorem 1. *If $r \leq t$, then for $\min(R(s-a), R(s-b), R(s-a-b)) > 1$,*

$$\begin{aligned} & \sum \phi_{r,\alpha,a}(n) \phi_{t,\beta,b}(n) n^{-s} \\ &= \zeta(s-a-b) \prod_p \left(1 + (-1)^\alpha \frac{p^{rb}}{p^{rs}} + (-1)^\beta \frac{p^{ta} + (-1)^\alpha p^{(t-r)a}}{p^{ts}} \right). \end{aligned}$$

PROOF. Since $\phi_{r,\alpha,a} = \mu_r^\alpha \star \zeta_a$, we have

$$\begin{aligned} \phi_{r,\alpha,a} \phi_{t,\beta,b} &= \zeta_{a+b} (\zeta_{-a} \mu_r^\alpha \star \zeta) (\zeta_{-b} \mu_t^\beta \star \zeta) \\ &= \zeta_{a+b} (h \star \zeta) = \zeta_{a+b} h \star \zeta_{a+b}, \end{aligned}$$

where $h = [\zeta_{-a} \mu_r^\alpha, \zeta_{-b} \mu_t^\beta]$. For all primes p and all $\gamma \geq 1$,

$$h(p^\gamma) = \sum_{\max(i,j)=\gamma} p^{-(ai+bj)} \mu_r(p^i)^\alpha \mu_t(p^j)^\beta.$$

Thus, $h(p^\gamma) = 0$ unless $\gamma = r$ or t . For $r < t$,

$$h(p^r) = (-1)^\alpha p^{-ra}, \quad (\zeta_{a+b} h)(p^r) = (-1)^\alpha p^{rb},$$

and

$$\begin{aligned} h(p^t) &= (-1)^\beta p^{-tb} + (-1)^{\alpha+\beta} p^{-(ra+tb)}, \\ (\zeta_{a+b} h)(p^t) &= (-1)^\beta p^{ta} + (-1)^{\alpha+\beta} p^{(t-r)a}. \end{aligned}$$

For $r = t$,

$$\begin{aligned} h(p^r) &= (-1)^\alpha p^{-ra} + (-1)^\beta p^{rb} + (-1)^{\alpha+\beta} p^{-r(a+b)}, \\ (\zeta_{a+b} h)(p^r) &= (-1)^\alpha p^{rb} + (-1)^\beta p^{ra} + (-1)^{\alpha+\beta}. \end{aligned}$$

The proof is completed by noting that in the statement of the theorem, the Dirichlet series on the left-hand side, and the series for the zeta function and the infinite product on the right-hand side, converge absolutely in the given half-plane.

Several examples of special interest: if a is a positive integer then for $R(s) > a + 2$,

$$\sum J_a(n) \Phi_t(n) n^{-s} = \zeta(s - a - 1) \prod_p \left(1 - \frac{p}{p^s} - \frac{p^{ta} - p^{(t-1)a}}{p^{ts}} \right);$$

if $r \leq t$ then for $R(s) > 3$,

$$\sum \Phi_r(n) \Phi_t(n) n^{-s} = \zeta(s - 2) \prod_p \left(1 - \frac{p^r}{p^{rs}} - \frac{p^t - p^{t-r}}{p^{ts}} \right);$$

for $R(s) > 3$,

$$\sum \psi(n)^2 n^{-s} = \zeta(s - 2) \prod_p \left(1 + \frac{2p + 1}{p^s} \right).$$

The next theorem is a generalization of a result of REDMOND and SIVARAMAKRISHNAN [11].

Theorem 2. *In a suitable half-plane and for t a positive integer,*

$$\sum \varrho_{a,r}(n) \phi_{tr,\beta,b}(n) n^{-s} = \zeta(s - a - b) \zeta(r(s - b)) \prod_p \left(1 + (-1)^\beta \frac{\sum_{k=0}^t p^{(t-k)ra}}{p^{trs}} - (-1)^\beta \frac{p^{rb} \sum_{k=0}^{t-1} p^{(t-k)ra}}{p^{(t+1)rs}} \right).$$

PROOF. Recall the definition of a Gegenbauer function $\varrho_{a,r}$ from Section 1,

$$\varrho_{a,r}(n) = \sum_{d|n} \nu_r(d) (n/d)^a \quad \text{for all } n,$$

where

$$\nu_r(n) = \begin{cases} 1 & \text{if } n \text{ is an } r\text{-th power} \\ 0 & \text{otherwise.} \end{cases}$$

The function ν_r is the r -th convolute of the arithmetical function ζ . Since $\mu_r^{-1} = \nu_r$, the function $\varrho_{a,r}$ is related to $\phi_{r,1,a}$ as the function $\sigma_a = \varrho_{a,1}$ is related to $\phi_a = \phi_{1,1,a}$.

We have

$$\begin{aligned}\varrho_{a,r}\phi_{tr,\beta,b} &= \zeta_{a+b}(\zeta_{-a}\nu_r \star \zeta)(\zeta_{-b}\mu_{tr}^\beta \star \zeta) = \\ &= \zeta_{a+b}(h \star \zeta) = \zeta_{a+b}h \star \zeta_{a+b},\end{aligned}$$

where $h = [\zeta_{-a}\nu_r, \zeta_{-b}\mu_{tr}^\beta]$. For all primes p and all $\gamma \geq 1$,

$$(\zeta_{a+b}h)(p^\gamma) = \sum_{\max(i,j)=\gamma} p^{(\gamma-i)a} p^{(\gamma-j)b} \nu_r(p^i) \mu_{tr}^\beta(p^j).$$

This is equal to zero unless γ is a multiple of r . For $1 \leq k \leq t-1$, $(\zeta_{a+b}h)(p^{kr}) = p^{krb}$,

$$(\zeta_{a+b}h)(p^{tr}) = p^{trb} + (-1)^\beta \sum_{k=0}^t p^{(t-k)ra}$$

and for $k > t$, $(\zeta_{a+b}h)(p^{kr}) = p^{(k-t)rb}(p^{trb} + (-1)^\beta)$. The Dirichlet series of the function $\zeta_{a+b}h$ is therefore, equal to

$$\begin{aligned}\prod_p \left(1 + \sum_{k=1}^{t-1} p^{-kr(s-b)} + \left(p^{trb} + (-1)^\beta \sum_{k=0}^t p^{(t-k)ra} \right) p^{-trs} \right. \\ \left. + \sum_{k=t+1}^{\infty} p^{(k-t)rb} (p^{trb} + (-1)^\beta) p^{-krs} \right)\end{aligned}$$

The Dirichlet series of $\varrho_{a,r}\phi_{tr,\beta,b}$ is $\zeta(s-a-b)$ times this expression, and after some manipulation, we obtain the right-hand side of the formula of the theorem.

With $r = 1$ and $\beta = b = 1$ we obtain

$$\begin{aligned}\sum \sigma_a(n) \Phi_t(n) n^{-s} = \\ = \zeta(s-a-1) \zeta(s-1) \prod_p \left(1 - \frac{\sum_{k=0}^t p^{(t-k)a}}{p^{ts}} + \frac{\sum_{k=0}^{t-1} p^{(t-k)a+1}}{p^{(t+1)s}} \right),\end{aligned}$$

and the generating function of $\sigma_a \Psi_t$ is obtained by changing the signs before the two fractional terms on the right-hand side. In particular, with $r = 1$, $t = 1$, $\beta = 2$ and $b = 1$,

$$\sum \sigma_a(n) \psi(n) n^{-s} = \zeta(s-a-1) \zeta(s-1) \prod_p \left(1 + \frac{p^a + 1}{p^s} - \frac{p^{a+1}}{p^{2s}} \right).$$

3. Kotelyanskii's theorem

Starting from a result of N. P. ROMANOV, which was obtained using Hilbert space methods, D. KOTELYANSKII [7] obtained a result which can be stated as follows.

Theorem 3. *If f and g are arithmetical functions and $h = [f, g]$, then*

$$(3) \quad \sum h(n)n^{-s} = \sum \phi_s(n)n^{-2s} \sum_{j=1}^{\infty} f(nj)j^{-s} \sum_{k=1}^{\infty} g(nk)k^{-s}$$

in any half plane in which the various Dirichlet series converge absolutely.

In fact, there is a short, simple proof of Theorem 3 which makes the result accessible to students in a beginning course in the theory of numbers. By (2), $h = (f \star \zeta)(g \star \zeta) \star \mu$. Hence, the left-hand side of (3) is equal to

$$\sum n^{-s} \sum_{d|n} \left(\sum_{a|d} f(a) \right) \left(\sum_{b|d} g(b) \right) \mu(n/d).$$

As a and b run independently over the positive integers, for each pair of values of a and b there is one term containing $f(a)g(b)$ as a factor for each divisor d of each multiple $[a, b]m$ of the least common multiple $[a, b]$ of a and b such that $[a, b]|d$. Thus, the quadruple sum is equal to

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} f(a)g(b) \sum_{m=1}^{\infty} [a, b]^{-s} m^{-s} \sum_{e|m} \mu(m/e),$$

and by the defining property of the Möbius function, this is equal to

$$(4) \quad \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} f(a)g(b)[a, b]^{-s},$$

On the other hand, the right-hand side of (3) is equal to

$$\begin{aligned} & \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} (f(a)a^{-s})(g(b)b^{-s}) \sum_{d|(a,b)} \phi_s(d) = \\ & = \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} f(a)g(b)a^{-s}b^{-s}(a, b)^s, \end{aligned}$$

which is equal to (4).

There is an unfortunate misprint on page 457 of [7], where the exponent on n in the right-hand side of (3) is s rather than $2s$: it is corrected in Mathematical Reviews, vol. 15, page 779.

KOTELYANSKIĬ used (3) to give a proof of Ramanujan's formula (1). We will use it to obtain a generalization of (1), a result even more general than the one due to Crum which is mentioned in Section 1.

When we began work on this paper, we used Kotelyanskii's result to prove Theorem 1 and 2. Then we noticed that the simple and direct proofs given in Section 2 allowed us to side step Theorem 3. In those proofs, $(\zeta_{a+b}h)(p^k)$ has a bounded number of terms for k large. However, if we set out to use the same method to find the generating function of the product $\varrho_{a,r}\varrho_{b,t}$, we find that we must consider $(\zeta_{a+b}h)(p^{km})$, where $m = [r, t]$ and $h = [\zeta_{-a}\nu_r, \zeta_{-b}\nu_t]$, and that it has an increasing number of terms for large, increasing values of k . This leads to a formidable problem of manipulation in simplifying the generating function. It is much easier to use Theorem 3 to prove the following theorem which, although it does not give the generating function of $\varrho_{a,r}\varrho_{b,t}$ for all r and t , does generalize Crum's result.

Theorem 4. For $\min(R(s), R(s-a), R(s-b), R(s-a-b)) > 1$, and t a positive integer,

$$\sum \varrho_{a,r}(n)\varrho_{b,tr}(n)n^{-s} = \zeta(s-a-b)\zeta(r(s-b))\zeta(tr(s-a))\zeta(trs) \\ \prod_p \left(1 - \frac{(p^{tra} - 1)p^{r(a+b)} + p^{rs}(p^{ra} - p^{tra})}{(p^{ra} - 1)p^{(t+1)rs}} \right)$$

PROOF. Since

$$\frac{\varrho_{a,r}(n)}{n^a} = \sum_{d|n} \frac{\nu_r(d)}{d^a} \quad \text{for all } n,$$

we have, from Theorem 3,

$$(5) \quad \sum \varrho_{a,r}(n)\varrho_{b,tr}(n)n^{-(s+a+b)} = \\ = \zeta(s) \sum_{n=1}^{\infty} \frac{\phi_s(n)}{n^{2s+a+b}} \sum_{j=1}^{\infty} \frac{\nu_r(nj)}{j^{s+a}} \sum_{k=1}^{\infty} \frac{\nu_{tr}(nk)}{k^{s+b}}.$$

For a positive integer n , let $Q_r(n)$ be the smallest positive integer such that $nQ_r(n)$ is an r -th power. If nj is an r -th power then $Q_r(n)|j$

and $j/Q_r(n)$ is an r -th power. Thus, $\nu_r(nj) = 1$ when and only when $nj = nQ_r(n)i^r$ for some i . Therefore,

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{\nu_r(nj)}{j^{s+a}} &= \sum_{i=1}^{\infty} \frac{1}{Q_r(n)^{s+a} i^{r(s+a)}} \\ &= \frac{1}{Q_r(n)^{s+a}} \sum_{i=1}^{\infty} \frac{1}{i^{r(s+a)}} = \frac{\zeta(r(s+a))}{Q_r(n)^{s+a}}. \end{aligned}$$

If we substitute for the two inner sums in the right-hand side of (5), and replace s by $s - a - b$, we obtain

$$\begin{aligned} &\sum \varrho_{a,r}(n) \varrho_{b,tr}(n) n^{-s} \\ &= \zeta(s-a-b) \zeta(r(s-b)) \zeta(tr(s-a)) \sum_{n=1}^{\infty} \frac{\phi_{s-a-b}(n)}{(nQ_r(n))^{s-b} (nQ_{tr}(n))^{s-a}}, \end{aligned}$$

which holds in the stated half-plane. If we denote the sum on the right-hand side by S , then in that half plane,

$$S = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\phi_{s-a-b}(p^j)}{(p^j Q_r(p^j))^{s-b} (p^j Q_{tr}(p^j))^{s-a}} \right).$$

Let S_p be the factor of S corresponding to the prime p . For $\alpha = 1, 2, \dots$, $(p^j Q_{tr}(p^j))^{s-a} = p^{\alpha tr(s-a)}$ for j in the interval $(\alpha-1)tr+1 \leq j \leq \alpha tr$, and for $\beta = 1, \dots, t$, $(p^j Q_r(p^j))^{s-b} = p^{((\alpha-1)t+\beta)r(s-b)}$ for j in the subinterval $((\alpha-1)t + \beta - 1)r + 1 \leq j \leq ((\alpha-1)t + \beta)r$. If we sum over the values of j in the subinterval, and then over the values of β , we find after some manipulation that

$$\begin{aligned} S_p &= 1 + \frac{p^{r(s-a-b)} - 1}{p^{r(s-a-b)}} \frac{p^{tra} - 1}{p^{ra} - 1} \sum_{\alpha=1}^{\infty} \frac{1}{p^{\alpha trs}} \\ &= \frac{1}{1 - p^{-trs}} \left(1 - \frac{(p^{tra} - 1)p^{r(a+b)} + p^{rs}(p^{ra} - p^{tra})}{(p^{ra} - 1)p^{(t+1)rs}} \right). \end{aligned}$$

Thus, we obtain the formula of the theorem.

When $t = 1$, Theorem 4 is Crum's result, and with $r = 1$ we have

$$\begin{aligned} &\sum \sigma_a(n) \varrho_{b,t}(n) n^{-s} = \\ &= \zeta(s-a-b) \zeta(s-b) \zeta(t(s-a)) \zeta(ts) \\ &\quad \prod_p \left(1 - \frac{(p^{ta} - 1)p^{a+b} + p^s(p^a - p^{ta})}{(p^a - 1)p^{(t+1)s}} \right). \end{aligned}$$

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