

Invariant submanifolds of a trans-Sasakian manifold

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S. TANNO [T] and K. YANO and S. ISHIHARA [YI] have proved that any invariant submanifold of a Sasakian manifold is minimal. The authors [YI] have obtained conditions for an invariant submanifold of a normal contact manifold to be totally geodesic in the case of codimension 2, and M. KON [K] have studied the case of codimension ≥ 2 . He also studied invariant submanifolds satisfying the condition $\tilde{R}(X, \xi)h = 0$.

The purpose of this paper is to show that similar results hold true for a more general class of manifolds, namely the class of trans-Sasakian manifolds, [O]. In §1 we recall definitions and some properties of almost contact metric manifolds. In §2 we prove that any invariant submanifold of a trans-Sasakian manifold is also trans-Sasakian and minimal. We obtain conditions for an invariant submanifold of a trans-Sasakian manifold to be totally geodesic. Finally, we study invariant submanifolds satisfying the condition $\tilde{R}(X, \xi)h = 0$ in a trans-Sasakian manifold.

1. Preliminaries

A $(2n + 1)$ -dimensional real differentiable manifold M of class C^∞ is said to have a (φ, ξ, η) -structure, or an almost contact structure, if it admits a field φ of endomorphisms of the tangent spaces, a vector field ξ , and a 1-form η satisfying

$$\eta(\xi) = 1,$$

$$\varphi^2 = -I + \eta \otimes \xi,$$

where I denotes the identity transformation, [B].

Denote by $\mathcal{X}(M)$ the Lie algebra of C^∞ vector fields on M . A (para-compact) manifold M with a (φ, ξ, η) -structure admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where $X, Y \in \mathcal{X}(M)$. Then M is said to have a (φ, ξ, η, g) -structure or an almost contact metric structure and g is called a compatible metric. The 2-form Φ on M defined by

$$\Phi(X, Y) = g(X, \varphi Y)$$

is called the fundamental 2-form of the almost contact metric structure. If ∇ is the Riemannian connection of g , then

$$(\nabla_X \eta)Y = g(Y, \nabla_X \xi),$$

$$(\nabla_X \Phi)(Y, Z) = g(Y, (\nabla_X \varphi)Z).$$

An almost contact metric structure (φ, ξ, η, g) is said to be normal ($|N|$) if $(\nabla_X \varphi)Y - (\nabla_{\varphi X} \varphi)Y + \eta(Y)\nabla_X \xi = 0$;
 α -Sasakian ($|\alpha S|$) if $(\nabla_X \varphi)Y = \alpha[g(X, Y)\xi - \eta(Y)X]$, $\alpha \in R$;
 α -Kenmotsu ($|\alpha K|$) if $(\nabla_X \varphi)Y = \alpha[g(\varphi X, Y)\xi - \eta(Y)\varphi X]$, $\alpha \in R$;
 Trans-Sasakian ($|tS|$) if $(\nabla_X \Phi)(Y, Z) = \frac{1}{2n}[(g(X, Y)\eta(Z) - g(X, Z)\eta(Y))\delta\Phi(\xi) + (g(X, \varphi Y)\eta(Z) - g(X, \varphi(Z)\eta(Y))\delta\eta]$;
 where δ denotes the coderivative on M .

Remark. If (φ, ξ, η, g) is trans-Sasakian and $\delta\Phi(\xi) = \delta\eta = 0$ then (φ, ξ, η, g) is cosymplectic, [B].

The structure (φ, ξ, η, g) is said to be Kenmotsu if it is 1-Kenmotsu, and Sasakian if it is 1-Sasakian. The relations among these structures are represented in the following diagram (where $\rightarrow$ denotes strict inclusion),

$$\begin{array}{ccc} |\alpha S| & \searrow & \\ |\alpha K| & \nearrow & |tS| \rightarrow |N|. \end{array}$$

If M is a trans-Sasakian manifold, then it is α -Sasakian if $\alpha = \frac{\delta\Phi(\xi)}{2n}$ and $\delta\eta = 0$, and it is α -Kenmotsu if $\alpha = \frac{\delta\eta}{2n}$ and $\delta\Phi(\xi) = 0$. It follows that if M is trans-Sasakian, then

$$(1.1) \quad \nabla_X \xi = -\frac{1}{2n}[(X - \eta(X)\xi)\delta\eta + \varphi X \delta\Phi(\xi)].$$

Moreover, the Riemann curvature tensor R satisfies the following property,

[ChG]:

(1.2)

$$\begin{aligned}
 R(X, Y, Z, W) = & R(X, Y, \varphi Z, \varphi W) \\
 & + \left(\left(\frac{\delta \Phi(\xi)}{2n} \right)^2 - \left(\frac{\delta \eta}{2n} \right)^2 \right) \{ -g(X, Z)g(Y, W) + g(X, W) \\
 & \quad g(Y, Z) + g(X, \varphi Z)g(Y, \varphi W) - g(X, \varphi W)g(Y, \varphi Z) \} \\
 & - \frac{1}{2n} [X(\delta \eta) \{ g(Y, Z)\eta(W) - g(Y, W)\eta(Z) \} - \\
 & \quad - Y(\delta \eta) \{ g(X, Z)\eta(W) - g(X, W)\eta(Z) \}] + \frac{1}{2n} [X(\delta \Phi(\xi)) \\
 & \quad \{ g(Y, \varphi W)\eta(Z) - g(Y, \varphi Z)\eta(W) \} - Y(\delta \Phi(\xi)) \\
 & \quad \{ g(X, \varphi W)\eta(Z) - g(X, \varphi Z)\eta(W) \}].
 \end{aligned}$$

For an extensive study of these structures we refer to [B], [ChG], [JV], [O].

2. Invariant submanifolds of a trans-Sasakian manifold

A $(2r + 1)$ -dimensional submanifold M of a $(2n + 1)$ -dimensional almost contact metric manifold $\tilde{M}$ with structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is said to be invariant if $\tilde{\varphi}X$ is tangent to M for any tangent vector field X to M , and $\tilde{\xi}$ is always tangent to M . It is well known that any invariant submanifold M of an almost contact metric manifold $\tilde{M}$ is also an almost contact metric manifold with the induced structure (φ, ξ, η, g) , where $\varphi X = \tilde{\varphi}X$, $X \in \mathcal{X}(M)$, ξ, η and g are the restrictions of $\tilde{\xi}, \tilde{\eta}$ and $\tilde{g}$ to M (see [YI]). Moreover, for any vector fields X, Y, Z on M , we have

$$(2.1) \quad \Phi(X, Y) = \tilde{\Phi}(X, Y),$$

$$(2.2) \quad (\tilde{\nabla}_X \tilde{\varphi})Y = (\nabla_X \varphi)Y + h(X, \varphi Y) - \tilde{\varphi}(h(X, Y)),$$

$$(2.3) \quad (\tilde{\nabla}_X \tilde{\Phi})(Y, Z) = (\nabla_X \Phi)(Y, Z),$$

$$(2.4) \quad (\tilde{\nabla}_X \tilde{\eta})Y = (\nabla_X \eta)Y,$$

where $\tilde{\Phi}$ and $\tilde{\nabla}$ (resp. Φ and ∇) denote the fundamental 2-form and the Riemannian connection in $\tilde{M}$ (resp. M), and h is the second fundamental form of M .

Let $\tilde{\delta}$ and δ denote the coderivatives on $\tilde{M}$ and M , respectively, and let $\mathcal{X}(M)^\perp$ denote the set of all vector fields orthogonal to M . Then from (2.3) and (2.4) we have

Lemma 2.1. *If M is an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$, then,*

$$(2.5) \quad \tilde{\delta}\tilde{\Phi}(X) = \frac{n}{r}\delta\Phi(X), \quad X \in \mathcal{X}(M),$$

$$(2.6) \quad \tilde{\delta}\tilde{\eta} = \frac{n}{r}\delta\eta.$$

Now, from (2.3) and lemma 2.1 we obtain

Proposition 2.1. *Any invariant submanifold M with induced structure (φ, ξ, η, g) of a trans-Sasakian manifold $\tilde{M}$ is also trans-Sasakian.*

Proposition 2.2. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then we have*

$$(2.7) \quad h(X, \varphi Y) = \tilde{\varphi}(h(X, Y)),$$

$$(2.8) \quad h(\varphi X, \varphi Y) = -h(X, Y),$$

$$(2.9) \quad h(X, \xi) = 0,$$

for any vector fields X and Y on M .

PROOF. Making use of the definition of trans-Sasakian manifold we obtain that $\tilde{\delta}\tilde{\Phi}(N) = 0$ for any $N \in \mathcal{X}(M)^\perp$, and moreover, from (2.2) we have

$$\tilde{g}((\tilde{\nabla}_X \tilde{\varphi})Y, N) = \tilde{g}(h(X, \varphi Y) - \tilde{\varphi}(h(X, Y)), N) = -\frac{1}{2n}\tilde{g}(X, Y)\tilde{\delta}\tilde{\Phi}(N) = 0,$$

for any $X, Y \in \mathcal{X}(M)$ and $N \in \mathcal{X}(M)^\perp$. Thus we obtain (2.7).

Also, from (2.7), we have $h(X, \varphi Y) = \tilde{\varphi}(h(X, Y)) = h(Y, \varphi X)$. Finally, using this relation we deduce (2.8) and (2.9).

From the identity (2.8) we conclude:

Theorem 2.1. *Any invariant submanifold M of a trans-Sasakian manifold $\tilde{M}$ is minimal.*

Corollary 2.1. *Any invariant submanifold M of an α -Sasakian, or of an α -Kenmotsu manifold $\tilde{M}$ is minimal.*

Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$ and R and $\tilde{R}$ be the Riemannian curvature tensors of M and $\tilde{M}$, respectively. Then, using the equation of Gauss, [KN], and proposition 2.2, we have

$$(2.10) \quad R(X, \varphi X, X, \varphi X) = \tilde{R}(X, \varphi X, X, \varphi X) - 2\tilde{g}(h(X, X)h(X, X)),$$

for any vector field X on M .

Let K (resp. $\tilde{K}$) be a φ (resp. $\tilde{\varphi}$) -sectional curvature of M (resp. $\tilde{M}$).

Then, from (2.10), we obtain

Proposition 2.3. *Let M be an invariant submanifold of $\tilde{M}$. Then the φ - and $\tilde{\varphi}$ -sectional curvatures satisfy the inequality $K \leq \tilde{K}$, with equality holding if and only if M is totally geodesic.*

Next, we shall give some conditions for an invariant submanifold of a trans-Sasakian manifold to be totally geodesic.

Proposition 2.4. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is totally geodesic if and only if $(\tilde{\nabla}_X \tilde{\nabla}_Y h)(\xi, \xi) = 0$ for any vector field X and Y on M .*

PROOF. From (1.1) we obtain that

$$\varphi X = -\frac{2n}{\delta\Phi^2(\xi) + \delta^2\eta} (\delta\Phi(\xi)\nabla_X \xi + \delta\eta\nabla_{\varphi X} \xi).$$

Thus, using the definition of the covariant derivative for the second fundamental form h of M , (see [KN], p.25), we have

$$\begin{aligned} h(X, Y) = -h(\varphi X, \varphi Y) = & -\frac{2n}{\delta\Phi^2(\xi) + \delta^2\eta} [\delta\Phi(\xi)(\tilde{\nabla}_Y h)(\varphi X, \xi) \\ & + \delta\eta(\tilde{\nabla}_{\varphi Y} h)(\varphi X, \xi)]. \end{aligned}$$

Now, using this relation and definition of the covariant derivative of h we obtain

$$\begin{aligned} h(\varphi X, \varphi Y) = & \frac{1}{2} \left(\frac{2n}{\delta\Phi^2(\xi) + \delta^2\eta} \right)^2 [\delta\Phi(\xi)^2 (\tilde{\nabla}_X \tilde{\nabla}_Y h)(\xi, \xi) + \delta\eta\delta\Phi(\xi) \\ & ((\tilde{\nabla}_{\varphi X} \tilde{\nabla}_Y h)(\xi, \xi) + (\tilde{\nabla}_X \tilde{\nabla}_{\varphi Y} h)(\xi, \xi)) + \delta\eta^2 (\tilde{\nabla}_{\varphi X} \tilde{\nabla}_{\varphi Y} h)(\xi, \xi)] \end{aligned}$$

which proves our assertion.

We now put

$$\tilde{R}(X, Y)\alpha = [\tilde{\nabla}_X, \tilde{\nabla}_Y]\alpha - \tilde{\nabla}_{[X, Y]}\alpha,$$

for a normal a bundle valued symmetric 2-form α . Then we have

$$\begin{aligned} (2.11) \quad (\tilde{R}(X, Y)h)(V, W) = & R^\perp(X, Y)(h(V, W)) - \\ & - h((R(X, Y)V, W) - h(V, R(X, Y)W), \end{aligned}$$

for any vector fields X, Y, V and W on M , where $R^\perp(X, Y) = [D_X, D_Y] - D_{[X, Y]}$, and D is the linear connection in the normal bundle $T(M)^\perp$.

Proposition 2.5. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then M is totally geodesic if and only if $\tilde{R}(X, \xi)h = 0$, for any vector field X on M .*

PROOF. If $\tilde{R}(X, \xi)h = 0$, from (2.11), we have

$$R^\perp(X, \xi)(h(V, W)) = h(R(X, \xi)V, W) + h(V, R(X, \xi)W)$$

Then, if we put $V = \xi$, from (2.9),

$$(2.12) \quad h(R(X, \xi)\xi, W) = 0.$$

On the other hand, since M is trans-Sasakian, using (1.2),

$$\begin{aligned} R(X, \xi, \xi, W) &= \left(\left(\frac{\delta\Phi(\xi)}{2n} \right)^2 - \left(\frac{\delta\eta}{2n} \right)^2 \right) \{g(X - \eta(X)\xi, W - \eta(W)\xi)\} \\ &\quad - \frac{1}{2n} \xi(\delta\eta)g(X - \eta(X)\xi, W - \eta(W)\xi) - \frac{1}{2n} \xi(\delta\Phi(\xi))g(X, \varphi W). \end{aligned}$$

Then, we have

$$\begin{aligned} (2.13) \quad R(X, \xi)\xi &= \left(\left(\frac{\delta\Phi(\xi)}{2n} \right)^2 - \left(\frac{\delta\eta}{2n} \right)^2 - \frac{1}{2n} \xi(\delta\eta) \right) \\ &\quad (X - \eta(X)\xi) + \frac{1}{2n} \xi(\delta\Phi(\xi))\varphi X. \end{aligned}$$

From (2.12) and (2.13), we obtain

$$\left(\left(\frac{\delta\Phi(\xi)}{2n} \right)^2 - \left(\frac{\delta\eta}{2n} \right)^2 - \frac{1}{2n} \xi(\delta\eta) \right) h(X, W) + \frac{1}{2n} \xi(\delta\Phi(\xi))h(\varphi X, W) = 0,$$

and, replacing X by φX , we deduce that $h(X, Y) = 0$, which shows that M is totally geodesic.

Finally, from propositions 2.4 and 2.5 we obtain

Theorem 2.2. *Let M be an invariant submanifold of a trans-Sasakian manifold $\tilde{M}$. Then, the following conditions are equivalent:*

- (i) M is totally geodesic;
- (ii) $(\tilde{\nabla}_X \tilde{\nabla}_Y h)(\xi, \xi) = 0$;
- (iii) $\tilde{R}(X, \xi)h = 0$;
- (iv) $\tilde{R}(X, Y)h = 0$,

where X and Y are arbitrary vector fields on M .

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