Publ. Math. Debrecen
38 / 1–2 (1991), 121–143

Splitting theorems for nonassociative rings

K. I. BEĬDAR (Moscow) and R. WIEGANDT (Budapest)

To the memory of Professor O. Steinfeld.

1. Introduction

In the algebra various direct decomposition theorems are known asserting that a certain kind of algebras decomposes into a direct sum of two (or more) uniquely determined distinguished subalgebras of diverse properties. Such theorems will be referred to as splitting theorems. In section 2 we shall discuss the general theory of splitting theorems in terms of Plotkin radicals. A natural Plotkin radical is introduced for alternative rings in section 3 and a splitting theorem is proved for semiprime alternative MPR–rings which generalizes a result of SLATER [19] and ZHEVLAKOV [24] (MPR means d.c.c. on principal right ideals). Section 4 is devoted to the splitting of the maximal torsion ideal of a not necessarily associative ring and two criteria are proved for the splitting. Applying the results of section 4, in section 5 it is proved that *every alternative MPR–ring splits with respect to the torsion radical.* This result generalizes the AYOUB – HUYNH Theorem [1], [7] on the splitting of associative MPR–rings and WIDIGER'S Theorem [23] of the splitting of alternative artinian rings. Finally a sufficient condition is given for the splitting of Jordan rings with respect to the maximal torsion ideal.

Next, as a motivation for the investigations we present a sample of splitting theorems.

1) If $(\mathcal{T},\mathcal{F})$ is a *centrally splitting torsion theory*, then every module is a direct sum of its maximal $\mathcal{T}$–module and $\mathcal{F}$–module, (cf. for instance [2], [12], [13]).

2) Every *associative artinian ring* is a direct sum of its maximal torsion ideal and of a uniquely determined torsionfree ideal ([21]). This statement is true also for *associative MPR–rings* ([1], [7]) and for *alternative artinian rings* ([23]).

3) Every *right and left artinian ring* is a direct sum of two uniquely determined ideals I and K such that $|I| < \alpha$ and $|A/K| < \alpha$ where α is an arbitrarily given infinite cardinal ([8]).

4) The Jacobson radical of every member of *an associative ring variety* is a direct summand if and only if the variety satisfies the identities

$$x^k y = y x^k = x^k y^n$$

for some integers $k \geq 1$ and $n \geq 2$ (see [22]). As M. V. VOLKOV has pointed out, also the Jacobson semisimple summand is a uniquely determined ideal (private communication).

5) Every *autodistributive ring* A (and algebra over the field of two elements) is a direct sum of the ideals

$$I = \{a \in A : a^2 = a\}$$

and

$$N = \{b \in A : b \cdot b^2 = 0\}$$

(cf. [10], [11], [14]).

6) Every *weakly distributive algebra* decomposes into a direct sum of three uniquely determined ideals which are a lattice ordered group (the multiplication defines the meet in the lattice) a commutative regular ring and an algebra of characteristic 3 (see [16]).

7) Let $\{P, Q\}$ be a partition of the prime numbers. Every *torsion abelian group* is a direct sum of its maximal P–subgroup and Q–subgroup, and these subgroups are uniquely determined.

8) In the variety of complemented semigroups satisfying the identities IR and IV of [4] every semigroup is a direct product of an idempotent semigroup and of a cancellative semigroup, and the factors are uniquely determined subsemigroups.

9) In the variety of *complemented semigroups* satisfying the identities IR and BV (or IV and BV) of [4] every semigroup is a direct product of two uniquely determined subsemigroups, one is a boolean ring, the other the "positive cone of an ℓ–group" (in German: Verbandsgruppenkern; cf. [4] Korollar 2 and Satz 15).

10) A "*divisibility semigroup*" (in German: Teilbarkeitshalbgruppe) satisfying the identity of [5] Satz 2, is a direct product of a distributive lattice and of a lattice–ordered group.

Let us observe that in many cases of the above examples the decomposition theorem holds only in a rather restricted subclass of the variety considered (e.g. artinian rings, or torsion abelian groups). There are also instances for decomposition theorems in which only one component is a uniquely determined subalgebra, we mention here two such examples.

11) Every *abelian group* is a direct sum of its maximal divisible subgroup and of *a* reduced subgroup.

12) Every *right group* is a direct product of its maximal right zero subsemigroup and of a group ([6]).

2. General theory

Let $\mathcal{V}$ be a variety of nonassociative rings (or Ω–groups, etc.). A subclass $\mathcal{A}$ of $\mathcal{V}$ is called a *universal class*, if $\mathcal{A}$ is closed under taking homomorphic images and ideals. A subclass ρ of a universal class $\mathcal{A}$ is called a *Plotkin radical* [17] (or pretorsion class in [3]), if ρ is homomorphically closed and every $A \in \mathcal{A}$ has a unique largest ρ–ideal ρA, called the *ρ–radical of A*. The class

$$\sigma = \{A \in \mathcal{A} : \rho A = 0\}$$

is referred to the *semisimple class of ρ*. Clearly $\rho \cap \sigma = 0$. A class $\gamma \subseteq \mathcal{V}$ is said to be *hereditary,* if it is closed under taking ideals. The *Kurosh–Amitsur radicals* are precisely those Plotkin radicals ρ which are *closed under extensions,* that is, $I \triangleleft A \in \mathcal{A}$, $I \in \rho$ and $A/I \in \rho$ imply $A \in \rho$, or equivalently, $\rho(A/\rho A) = 0$ for all $A \in \mathcal{A}$. The semisimple class σ of a Kurosh–Amitsur radical ρ is always *regular,* that is, if $0 \neq I \triangleleft A \in \sigma$, then I has a nonzero homomorphic image in σ.

Proposition 1. *Let ρ be a hereditary Plotkin radical in a universal class $\mathcal{A} \subseteq \mathcal{V}$ and σ its semisimple class. The ρ–radical ρA of A is a direct summand of A if and only if there exists an ideal K of A such that $K \in \sigma$ and $A = \rho A + K$.*

The necessity is obvious. For the sufficiency we have to prove only that $\rho A \cap K = 0$. Since ρ is hereditary, $\rho A \cap K \triangleleft \rho A \in \rho$ implies $\rho A \cap K \in \rho$. Hence by $\rho A \cap K \triangleleft K \in \sigma$ it follows $\rho A \cap K \subseteq \rho K = 0$, and so $A = \rho A \boxplus K$ is a direct sum.

We say that an object $A \in \mathcal{A}$ *splits with respect to the radical ρ* if A is a direct sum $A = \rho A \boxplus \sigma A$ such that σA is the unique largest σ–ideal of A. In this case we also say that the *ρ–radical ρA of A splits off.*

Theorem 1. *Let ρ be a hereditary Plotkin radical in a universal class $\mathcal{A} \subseteq \mathcal{V}$ and σ the corresponding semisimple class. An object $A \in \mathcal{A}$ splits with respect to ρ if and only if*

1) there exists an ideal K of A such that $K \in \sigma$ and $A = \rho A + K$,

2) for any two maximal σ–ideals I and L of A $\quad I/(I \cap L) \in \sigma$.

PROOF. *Necessity.* 1) follows from Proposition 1 and 2) becomes trivial.

Sufficiency. First we prove that K is a maximal σ–ideal of A. Suppose that an ideal M properly contains K. Then in view of Proposition 1 we have $A = \rho A \boxplus K$, and so it follows $M = R \boxplus K$ with a suitable ideal $0 \neq R \subseteq \rho A$. Since ρ is hereditary, we have $R \in \rho$, and therefore $0 \neq R \subseteq \rho M$, that is, $M \notin \sigma$. Hence K is a maximal σ–ideal of A. Let I be any other maximal σ–ideal of A. Now we have

$$I/(I \cap K) \simeq (I + K)/K \triangleleft A/K \simeq \rho A \in \rho.$$

Hence condition 2), the hereditariness of ρ and $\rho \cap \sigma = 0$ yield $I = K$.

Proposition 2. *If ρ is a hereditary Plotkin radical in $\mathcal{A}$ with semisimple class σ, then every object $A \in \mathcal{A}$ has maximal σ–ideals .*

PROOF. Let $I_1 \subseteq \ldots \subseteq I_\alpha \subseteq \ldots$ be an ascending chain of ideals of $A \in \mathcal{A}$ such that $I_\alpha \in \sigma$ for each index α. Putting $I = \cup I_\alpha$, we have

$$I_\alpha \cap \rho I \subseteq \rho I \in \rho,$$

and so the hereditariness of ρ implies $I_\alpha \cap \rho I \in \rho$ for all indices α. Hence $(I_\alpha \cap \rho I) \subseteq \rho I_\alpha = 0$ for all α, yielding $\rho I = 0$, that is, $I \in \sigma$. Now an application of Zorn's Lemma yields the assertion.

Proposition 3. *The semisimple class σ of a Plotkin radical ρ is closed under extensions.*

PROOF. Since ρ is homomorphically closed, for every ideal I of $A \in \mathcal{A}$ the relation

$$\rho A/(\rho A \cap I) \simeq (\rho A + I)/I \subseteq \rho(A/I)$$

must be true. Hence, if $A/I \in \sigma$, we get $\rho A \subseteq I$. If also $I \in \sigma$ holds, then by $\rho A \in \rho$ it follows $\rho A \subseteq \rho I = 0$, that is, $A \in \sigma$.

Proposition 4. *Let ρ be a Plotkin radical with semisimple class σ. If every $A \in \mathcal{A}$ is a direct sum $A = \rho A \boxplus K$ with an ideal $K \in \sigma$, then ρ is a Kurosh–Amitsur radical in $\mathcal{A}$.*

PROOF. For every A we have $A/\rho A \simeq K \in \sigma$, and so $\rho(A/\rho A) = 0$ holds.

Proposition 5. *If σ is a homomorphically closed class, then condition 2) of Theorem 1 is trivially fulfilled.*

Proposition 6. *If ρ is a hereditary Plotkin radical in $\mathcal{A}$ and its semisimple class σ is homomorphically closed, then every $A \in \mathcal{A}$ has a unique largest σ–ideal, and σ is a Kurosh–Amitsur radical class.*

Let us observe that if σ is a Plotkin semisimple class which is also a Plotkin radical, then σ is a Kurosh–Amitsur radical.

PROOF. Let I be an arbitrary σ–ideal of A. By Proposition 2 A has a maximal σ–ideal K. Since σ is homomorphically closed, by $I \in \sigma$ we get

$$(I + K)/K \simeq I/(I \cap K) \in \sigma.$$

Hence by Proposition 3 it follows $(I+K) \in \sigma$, and therefore the maximality of K implies $I \subseteq K$.

Now the second assertion follows immediately from Proposition 3.

In his [11] Theorem 2.1 Gardner proved the equivalence of the following statements:

1) in the variety $\mathcal{V}\rho$ is a radical semisimple class such that its semisimple class σ is a radical class with semisimple class ρ,

2) ρ and σ are subvarieties and every $A \in \mathcal{V}$ can be decomposed as a direct sum $A = R \boxplus S$ where $R \in \rho$ and $S \in \rho$ are uniquely determined ideals.

Our next result is a counterpart of Gardner's Theorem for universal classes.

Theorem 2. *Let ρ be a hereditary Plotkin radical with semisimple class σ in a universal class $\mathcal{A} \subseteq \mathcal{V}$. Then the following conditions are equivalent:*

I) σ is a (Plotkin) radical with semisimple class ρ,

II) σ is homomorphically closed and every $A \in \mathcal{A}$ splits with respect to ρ.

PROOF. $I \Rightarrow II$. Since both ρ and σ are Kurosh–Amitsur radicals by Proposition 3, and σ is homomorphically closed, we have

$$A/(\rho A + \sigma A) \simeq \frac{A/\rho A}{(\rho A + \sigma A)/\rho A} \in \sigma$$

and

$$A/(\rho A + \sigma A) \simeq \frac{A/\sigma A}{(\rho A + \sigma A)/\sigma A} \in \rho$$

which implies $A/(\rho A + \sigma A) = 0$, that is, $A = \rho A + \sigma A$. Hence by Proposition 5 we can apply Theorem 1 establishing II.

$II \Rightarrow I$. By Proposition 6 σ is a Kurosh–Amitsur radical class. If $\sigma A = 0$, then by the unique decomposition $A = \rho A \boxplus \sigma A$ it follows $A = \rho A \in \rho$, and conversely. Hence

$$\rho = \{A \in \mathcal{A} : \sigma A = 0\}$$

is the semisimple class of σ.

In the variety $\mathcal{V}$ of all associative rings every radical semisimple class $\rho \neq \mathcal{V}$ consists of idempotent rings and so only the radicals $\rho = 0$ and $\rho = \mathcal{V}$ satisfy condition I in Theorem 2.

3. Alternative rings and the maximal nuclear ideal

In view of Proposition 4 one gets the impression that in the reasonable cases the Plotkin radical ρ has to be a Kurosh–Amitsur radical. This is, however, not so, and in this section we shall give an example for a very natural Plotkin radical in the variety $\mathcal{V}$ of *all alternative rings* which splits off in the subclass $\mathcal{B} \subseteq \mathcal{V}$ of semiprime MPR–rings. The class $\mathcal{B}$ is, of course, not a universal class (cf. Proposition 2 and Theorem 2).

A not necessarily associative ring A will be called an *MPR–ring*, if A satisfies the d.c.c. on principal right ideals. As is well–known, a ring A is *alternative*, if A satisfies

$$x^2y = x(xy) \quad \text{and} \quad (xy)y = xy^2$$

for all $x, y \in A$. The associator (x, y, z) of the elements $x, y, z \in A$ is defined by

$$(x, y, z) = (xy)z - x(yz).$$

An element $u \in A$ is said to be ***nuclear***, if the associator (u,x,y) vanishes for all $x,y \in A$. The set $\mathcal{N}(A)$ of all nuclear elements of A is the ***nucleus*** of A. A has a unique maximal ideal $\mathcal{U}(A)$ in $\mathcal{N}(A)$ which is called the ***maximal nuclear ideal*** of A.

Every alternative ring A possesses another distinguished ideal, the ***associator ideal*** $\mathcal{D}(A)$, which is the ideal of A generated by all associators. As is well–known $\mathcal{U}(A)\mathcal{D}(A) = \mathcal{D}(A)\mathcal{U}(A) = 0$. For details we refer to [25].

As in the associative case, an alternative ring A is said to be ***semiprime***, if $I \triangleleft A$ and $I^2 = 0$ imply $I = 0$.

Propositions 7. *If M is a minimal right ideal of a semiprime alternative ring A, then $M^2 \neq 0, M^2 = M$ and $M = eA$ for a nuclear idempotent e of A.*

PROOF. The first assertion follows immediately from [18] Lemma (3.3) which states that a trivial right ideal generates a trivial ideal. The rest is [20] Proposition 3.3 (b) and (c).

The sum of all minimal right ideals of a ring A is called the ***right socle*** of A and will be denoted by Soc A.

Proposition 8. *If A is an alternative semiprime MPR–ring, then $A =$ Soc A.*

PROOF. Let $a \neq 0$ be an arbitrary element of A. We shall prove that a is contained in a finite sum of minimal right ideals. Since A is an MPR–ring, there exists a minimal right ideal M_1 of A which is contained in the principal right ideal $[a)$ of A. By Proposition 7 there exists a nuclear idempotent e_1 such that $M_1 = e_1A$. Take the element $a_2 = a - e_1a$. We have obviously

$$[a) = M_1 + [a_2).$$

Let $x \in M_1 \cap [a_2)$ be an arbitrary element. Now x has the forms

$$x = e_1b, \qquad b \in A$$

and

$$x = ma_2 + \sum_{i=1}^{n}(\ldots((a_2c_{i1})c_{i2})\ldots)c_{ik_i}$$

where m is an integer and $c_{ij_i} \in A,\ j_i = 1,\ldots,k_i$. Since e_1 is a nuclear idempotent, we have

$$e_1x = e_1(e_1b) = e_1^2b = x$$

and

$$e_1a_2 = e_1a - e_1(e_1a) = e_1a - e_1^2a = 0,$$

furthermore, also

$$x = e_1 x = e_1(ma_2) + e_1 \sum_{i=1}^{n} (\ldots((a_2 c_{i1})c_{i2})\ldots)c_{ik_i} =$$
$$= m(e_1 a_2) + \sum_{i=1}^{n} (\ldots(((e_1 a_2)c_{i1})c_{i2})\ldots c_{ik_i}) = 0.$$

Thus $[a)$ is a direct sum

$$[a) = M_1 \oplus [a_2)$$

of the right ideals, and so $[a)$ properly contains a_2. Continuing this procedure we get a decomposition

$$[a) = M_1 \oplus \ldots \oplus M_{n-1} \oplus [a_n)$$

and also a strictly descending chain

$$[a) \supset [a_2) \supset \ldots \supset [a_n)$$

of principal right ideals of A. Since A is an MPR–ring, this chain has to terminate in finitely many steps. Thus $[a)$ is a finite direct sum of minimal right ideals, proving the assertion.

Proposition 9. (cf. [20] Proposition 4.15). *If A is an alternative semiprime ring, then*

$$\text{Soc } A = \text{Soc } \mathcal{U}(A) \boxplus \text{Soc } \mathcal{D}(A)$$

where Soc $\mathcal{U}(A)$ *and* Soc $\mathcal{D}(A)$ *are two–sided ideals in A.*

Theorem 3. *If A is an alternative semiprime MPR–ring, then the maximal nuclear ideal $\mathcal{U}(A)$ splits off, in fact*

$$A = \mathcal{U}(A) \boxplus \mathcal{D}(A).$$

PROOF. Since A is semiprime, $(\mathcal{U}(A) \cap \mathcal{D}(A))^2 \subseteq \mathcal{U}(A)\mathcal{D}(A) = 0$ implies $\mathcal{U}(A) \cap \mathcal{D}(A) = 0$. Hence Propositions 8 and 9 yield

$$A = \text{Soc } A = \text{Soc } \mathcal{U}(A) \boxplus \text{Soc } \mathcal{D}(A) \subseteq \mathcal{U}(A) \boxplus \mathcal{D}(A) \subseteq A.$$

Theorem 3 provides also a full description of the structure of any alternative semiprime MPR–ring A inasmuch as it reduces the description of A to that of an associative semiprime MPR–ring $\mathcal{U}(A)$ and to that of a purely alternative semiprime MPR–ring $\mathcal{D}(A)$. Hence $\mathcal{U}(A)$ is a discrete direct sum of simple rings of linear transformations of finite rank on vector spaces over division rings (cf. e.g. [15] Theorems 77.4 and 78.2 or [20], and $\mathcal{D}(A)$ is a discrete direct sum of Cayley–Dickson algebras (cf. [20])). Moreover, the right socle of an alternative semiprime MPR–ring coincides with its left socle. Thus Theorem 3 generalizes the ZHEVLAKOV – SLATER Theorem describing the structure of alternative semiprime artinian rings ([24] Theorem 3 and [19] Theorem B, or [25] Theorem 12.2.3).

As is well–known, semiprimeness is a hereditary property (Slater [18] or [25] Theorem 9.1.4). Hence the class $\beta = \{A \in \mathcal{V} \mid A$ has only 0 semiprime homomorphic image $\}$ is a Kurosh–Amitsur radical class, called the *Baer radical class* (cf. [25] p 162). Let us consider the class $\alpha = \{A \in \mathcal{V} \mid A/\beta(A)$ is associative $\}$. Since $\mathcal{N}(I) = I \cap \mathcal{N}(A)$ holds for every $I \triangleleft A$ (see [18] or [24] Theorem 9.1.1) by the heredity of semiprimeness it follows that *α is a hereditary Plotkin radical, but not a Kurosh–Amitsur radical in $\mathcal{V}$*. Furthermore, we have obviously

$$\alpha(A)/\beta(A) = \mathcal{U}(A/\beta(A)),$$

and hence $\alpha(A) = \mathcal{U}(A)$ for every semiprime ring A. Thus Theorem 3 yields immediately the following reformulation.

Corollary 1. *The Plotkin radical α splits off on the class $\mathcal{B}$ of all alternative semiprime MPR–rings.*

Let $\mathcal{J}$ denote the *quasi–regular radical* (or *Zhevlakov radical* in [25] Theorem 10.4.5).

Corollary 2. *If A is an alternative MPR–ring, then $\beta(A) = \mathcal{J}(A)$.*

PROOF. Clearly $\beta(A) \subseteq \mathcal{J}(A)$ always holds. Hence without loss of generality we may assume that $\beta(A) = 0$. Since a quasi–regular ideal does not contain nonzero idempotents, the assertion follows from Theorem 3.

4. Rings and the torsion radical

In this section we shall work in the variety $\mathcal{V}$ of *all not necessarily associative rings*. It is well–known that in $\mathcal{V}$ the class of all torsion rings forms a hereditary (Kurosh–Amitsur) radical class, and so we can speak of the *maximal torsion ideal* of a ring A. We know also that the maximal divisible subgroup of any ring $A \in \mathcal{V}$ forms an ideal, the *maximal divisible ideal* of A.

Proposition 10. *If F_+ is a torsionfree divisible subgroup and T_+ is a torsion subgroup of the additive group A_+ of a ring A, then $FT = TF = 0$.*

For the proof, which uses only the distributivity, we refer to Kertész [15] Proposition 57.7.

Proposition 11. *Let A be a ring such that $A_+ = B_+ \oplus D_+$ is a group direct sum of a reduced torsion group B_+ and of the maximal divisible subgroup D_+. Then*

1) $A_+ = T_+ \oplus F_+$ where T is the maximal torsion ideal of A and F_+ is a torsionfree divisible subgroup of A_+,

2) every quasi–cyclic subgroup of A_+ is in the annihilator of A,

3) $BD = DB = 0$,

4) $D/(T \cap D) \simeq A/T$.

PROOF. By $A_+ = B_+ \oplus D_+$ we have $T_+ = B_+ \oplus (T \cap D)_+$ and $D_+ = (T \cap D)_+ \oplus F_+$. Hence 1) is obviously true.

For the proof of 2) we refer to Kertész [15] Proposition 57.8. In view of the decomposition $D_+ = (T \cap D)_+ \oplus F_+$ statement 2) and Proposition 10 yield statement 3).

4) and its proof is Ayoub's Theorem 2.(3) in [1] where no associativity of the multiplication is used.

Theorem 4. *The following conditions are equivalent:*

$$\begin{aligned} I) &\qquad A = T \boxplus K, \\ II) &\qquad D^2 \textit{ is torsionfree}, \\ III) &\qquad F^2 \textit{ is torsionfree.} \end{aligned}$$

PROOF. By the assumption on A_+ we have

$$D^2 = ((T \cap D) + F)^2 = (T \cap D)^2 + (T \cap D)F + F(T \cap D) + F^2.$$

In view of Proposition 10 and Proposition 11, 2) the first three terms of the right hand side are 0. Hence $D^2 = F^2$ always holds proving the equivalence of II and III.

$I \Rightarrow II$. Let $A = T \boxplus K$ where K is a torsionfree ideal of A. By the structure of A_+ it follows that $D_+ = (T \cap D)_+ \oplus K_+$. Hence by the previous consideration we have $D^2 = K^2 \subseteq K$, and therefore D^2 is torsionfree.

$II \Rightarrow I$. By the distributivity D^2 is clearly a divisible group, hence $D_+ = (T \cap D)_+ \oplus D^2_+ \oplus C_+$ holds with a suitable divisible torsionfree subgroup C_+. Moreover, also $A_+ = T_+ \oplus D^2_+ \oplus C_+$ holds. By Proposition 10 we get

$$(D^2 + C)A = D^2(B + D) + C(B + D) = D^2B + D^2D + CB + CD \subseteq D^2,$$

and similarly also $A(D^2 + C) \subseteq D^2$. Thus $D^2 + C$ is an ideal in A, and so in the direct decomposition of A_+ both summands are ideals of A.

It may happen that *in the direct decomposition $A = T \boxplus F$ the ideal F* is not uniquely determined. For instance, let A be a ring such that $A^2 = 0$

and $A_+ = C(p^\infty) \oplus Q_+$ where $C(p^\infty)$ is a quasi–cyclic group and Q_+ is the additive group of rational numbers. Let $a_1, \dots, a_n, \dots$ be generators of $C(p^\infty)$ subjected to $pa_1 = 0, pa_2 = a_1, \dots, pa_n = a_{n-1}, \dots$. Let us define a mapping $\phi : Q_+ \to Q_+ \oplus C(p^\infty)$ by

$$\phi\left(\frac{k}{\ell}p^r\right) = \begin{cases} \dfrac{k}{\ell}p^r & \text{if } r \geq 0 \text{ (including } k = 0) \\ \dfrac{k}{\ell}p^r + \dfrac{k}{\ell}a_{-r} & \text{if } r < 0 \end{cases}$$

where $k, \ell \neq 0$, and p are mutually relatively prime. It is easy to check that ϕ is an isomorphism onto a subgroup $Q'_+ \neq Q_+, A_+ = C(p^\infty) \oplus Q'_+$ and so the second direct component of A is not uniquely determined as an ideal.

This example is typical, as we see it from the following

Proposition 12. *Let A be a ring such that $A_+ = B_+ \oplus D_+$ where B_+ is a reduced group and D_+ is a divisible group. If A_+ does not contain a quasi–cyclic subgroup, then the maximal torsion ideal T of A splits off.*

PROOF. By the assumption we have $B_+ = T_+$. Since both T and D are uniquely determined ideals, the assertion follows.

The smallest nonzero ideal of a subdirectly irreducible ring A is called the *heart* of A. Let $\mathcal{M}$ be a homomorphically closed subclass of $\mathcal{M}$ such that for every $A \in \mathcal{M}$, it holds $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion group and D_+ is the maximal divisible subgroup.

Theorem 5. *The following conditions are equivalent:*

I) for every ring $A \in \mathcal{M}$ the maximal torsion ideal T splits off,

II) for every ring $A \in \mathcal{M}$ D^2 is torsionfree where D denotes the maximal divisible ideal of A,

III) every subdirectly irreducible ring $A \in \mathcal{M}$ having a torsion heart, is a torsion ring.

PROOF. Theorems 1 and 4 yield the equivalence of I and II.

$I \Rightarrow III$. Let A be a subdirectly irreducible ring from the class $\mathcal{M}$ with torsion heart H. By I we have

$$A = T \boxplus F,$$

and hence also

$$F \cap H \subseteq F \cap T = 0.$$

Thus $F = 0$ and $A = T$ follows.

$III \Rightarrow II$. Assume that there are elements $a_1, \dots, a_n$, $b_1, \dots, b_n$ in the maximal divisible ideal D of a ring A such that $0 \neq \sum_{i=1}^{n} a_i b_i \in T$. Using Zorn's Lemma we can choose an ideal I of A such that I is maximal with respect to $\sum_{i=1}^{n} a_i b_i \notin I$. The ring $\overline{A} = A/I$ is clearly subdirectly irreducible, and its heart $\overline{H}$ contains the coset $\sum_{i=1}^{n} \overline{a}_i \overline{b}_i$ which is a torsion elément. Hence $\overline{H}$ has to be a torsion ring (in fact a p-torsion ring for some prime p.) Thus by III also the ring $\overline{A}$ is a torsion ring, and consequently its maximal divisible ideal $\overline{D}$ is the sum of quasi-cyclic groups and so $\overline{D}^2 = 0$. Since $\overline{a}_1, \dots, \overline{a}_n$ and $\overline{b}_1, \dots, \overline{b}_n$ are in $\overline{D}$, it follows $\sum_{i=1}^{n} \overline{a}_i \overline{b}_i = 0$ which contradicts $\sum_{i=1}^{n} \overline{a}_i \overline{b}_i \notin I$. Thus D^2 is torsionfree.

We shall make use of the following nearly trivial

Proposition 13. *If $[a_1, \dots, a_n]$ is any product of n elements and i is any of the integers $1, \dots, n$, then*

$$k[a_1, \dots, a_n] = [a_1, \dots, ka_i, \dots, a_n]$$

for every integer k. Moreover $[ka) = k[a)$ holds for the principal right ideals $[ka)$ and $[a), a \in A$.

PROOF. By the distributivity we have

$$(ka)b = k(ab) = a(kb)$$

for every element $a, b \in A \in \mathcal{A}$, and the assertion follows by induction. The rest is straightforward.

Proposition 14. *If A is an MPR–ring, then its additive group A_+ is a direct sum $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion group and D_+ is the maximal divisible subgroup of A_+.*

PROOF. The additive group of A can be surely decomposed as $A_+ = B_+ \oplus D_+$ where D_+ is the maximal divisible subgroup in A_+ and B_+ is a reduced group. Let us observe that B_+ does not contain nonzero divisible element. Since A is an MPR–ring, the set

$$\{[b) : b \in B \text{ and } o(b) = \infty\}$$

of principal right ideals of A has a minimal element, say $[c)$. Now we have

$$[c) = [2c) = \dots = [nc) = \dots .$$

Since by Proposition 13

$$[c) = [nc) = n[c),$$

the element c is divisible. Thus by $c \in B$ we conclude that $c = 0$, and so B is a torsion group.

Proposition 15. *If A is a torsionfree MPR–ring, then every element $a \in A$ is a finite sum*

$$a = \sum_{i=1}^{r} (\dots((ab_{i2})b_{i3})\dots)b_{is_i}$$

of products of at least two factors.

PROOF. Suppose that A is an MPR–ring. Then every descending chain

$$[a) \supset \dots \supset [p^k a) \supset \dots$$

terminates at an integer $k \geq 1$ for every prime number p. Hence

$$p^k a = \ell(p^{k+1}a) + \sum_{i=1}^{r} [p^{k+1}a, c_{i2}, \dots, c_{is_i}]$$

where ℓ is an integer $s_i \geq 2$ and $[x_1, \dots, x_n]$ stands for $(\dots((x_1x_2)x_3)\dots)x_n$. Further, for $n = p^k - \ell p^{k+1} \neq 0$ we have

$$na = \sum_{i=1}^{r} [p^{k+1}a, c_{i2}, \dots, c_{is_i}].$$

Since by Proposition 14 the additive group A_+ is divisible, to each c_{i2} there exists an element $d_{i2} \in A_+$ such that $nd_{i2} = c_{i2}$, $(i = 1, \dots, r)$. Hence by Proposition 13

$$na = n \sum_{i=1}^{r} [p^{k+1}a, d_{i2}, c_{i3}, \dots, c_{is_i}]$$

(if $s_i = 2$, then the product is just $p^{k+1}a \cdot d_{i2}$). Taking into account that A is torsionfree, we get

$$a = \sum_{i=1}^{r} [p^{k+1}a, d_{i2}, c_{i3}, \dots, c_{is_i}]$$

and hence Proposition 13 yields the desired form

$$a = \sum_{i=1}^{r} [a, p^{k+1}d_{i2}, c_{i3}, \dots, c_{is_i}]$$

and also the parantheses are in the required order.

Proposition 16. *The class of all torsionfree MPR–rings is homomorphically closed.*

PROOF. Let A be a torsionfree MPR–ring and $I \triangleleft A, a \in A$. Suppose that there exists an integer $k \neq 0$ such that $ka \in I$. By Proposition 15 we have

$$a = \sum_{i=1}^{r} [a, b_{i2}, \dots , b_{is_i}]$$

where $s_i \geq 2$ for each $i = 1, \dots , r$. Since by Proposition 11, 1) and 14 A_+ is divisible, for every b_{i2} there exists an element c_{i2} such that $kc_{i2} = b_{i2}, (i = 1, \dots , r)$. Now by Proposition 13 we have

$$a = \sum_{i=1}^{r} [a, kc_{i2}, b_{i3}, \dots , b_{is_i}] =$$
$$= \sum_{i=1}^{r} [ka, c_{i2}, b_{i3}, \dots , b_{is_i}] \in I.$$

Hence $ka \in I$ implies $a \in I$. Thus the factor ring A/I is torsionfree, and the assertion is proved.

Theorem 6. *For a not necessarily associative MPR–ring A the following conditions are equivalent:*

I) the maximal torsion ideal T of A splits off,

II) D^2 is torsionfree where D denotes the maximal divisible ideal of A,

III) if a subdirectly irreducible factor ring $\overline{A}$ of A has a torsion heart, then $\overline{A}$ is a torsion ring.

PROOF. By Propositions 14 and 16 Theorem 5 is applicable yielding the equivalences.

In the previous considerations we followed ideas of CHRISTINE W. AYOUB who proved that every associative MPR–ring splits with respect to the torsion ideal (cf. [1] Corollary 1). In fact, she proved in [1] Theorem 4 that D^2 is torsionfree.

5. Alternative rings and the torsion radical

In this section we shall work again in the variety $\mathcal{V}$ of *all alternative rings*. The main goal of this section is to prove that every alternative MPR–ring splits with respect to its maximal torsion ideal. In proving this, we shall use Theorems 3 and 6 as well as several arguments from the book [25].

Proposition 17. ([9] Lemma 1). *If M is a right ideal of an alternative ring A, then $AM + M \triangleleft A$.*

Proposition 18. *Let A be an alternative ring with maximal divisible ideal D and maximal torsion ideal T such that $A = D + T$. Further, let $a_1, \dots, a_n \in D$ and $b_1, \dots, b_n \in A$ be elements such that $\sum_{i=1}^{n} a_i b_i \in T$. If there exists elements $d \in A$ and $t_1, \dots, t_n \in T$ such that $b_i = b_i d + t_i$ for $i = 1, \dots, n$, then $\sum_{i=1}^{n} a_i b_i = 0$.*

PROOF. We shall use the Moufang identity

$$(xy)(zx) = x(yz)x \tag{1}$$

(see for instance [25] Lemma 2.3.7.).

In view of Propositions 10 and 11 it follows $TD = DT = 0$. Hence by $A = T + D$ we have

$$(T \cap D)A = A(T \cap D) = 0, \tag{2}$$

and so also

$$a_i t_i = 0 = (da_i)t_i, \qquad i = 1, \dots, n.$$

Using repeatedly (2) and finally (1) we get

$$\begin{aligned}
&\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} a_i(b_i d + t_i) = \sum_{i=1}^{n} a_i(b_i d) = \\
&= \sum_{i=1}^{n} a_i(b_i d) - \left(\sum_{i=1}^{n} a_i b_i\right) d = -\sum_{i=1}^{n}(a_i, b_i, d) = -\sum_{i=1}^{n}(d, a_i, b_i) = \\
&= -\sum_{i=1}^{n}(da_i)b_i + d\left(\sum_{i=1}^{n} a_i b_i\right) = -\sum_{i=1}^{n}(da_i)(b_i d + t_i) = \\
&= \sum_{i=1}^{n}(da_i)(b_i d) = -d\left(\sum_{i=1}^{n} a_i b_i\right) d = 0.
\end{aligned}$$

Proposition 19. *Again, let $A = D + T$ be an alternative ring, and let M be a right ideal of A such that $M \subseteq D$. If for any finitely many elements $m_1, \dots, m_n \in M$ there exists an idempotent $e \in A$ such that $m_i e = m_i$; $i = 1, \dots, n$, then $(AM + M) \cap T = 0$.*

PROOF. Let $m + \sum_{i=2}^{n} a_i m_i$ be an arbitrary element of $AM + M$, and suppose that this element is in T. Now there exists an idempotent $e \in A$

such that $me = m$ and $m_i e = m_i$ for $i = 2, \dots, n$. Since $A = D + T$, we have $e = x + y$ where $x \in T$ and $y \in D$. By (2) it follows $mx \in DT = 0$, and so

$$m = me = mx + my = my.$$

Putting $a_1 = m$ and $m_1 = y$ we get

$$\begin{aligned} m + \sum_{i=2}^{n} a_i m_i &= \sum_{i=1}^{n} a_i m_i = \sum_{i=1}^{n} a_i (m_i e) = \\ &= \left(\sum_{i=1}^{n} a_i m_i \right) e - \sum_{i=1}^{n} (a_i, m_i, e). \end{aligned}$$

Taking into account (2) and $m + \sum_{i=2}^{n} a_i m_i \in D \cap T$, and (1) we may continue

$$\begin{aligned} m + \sum_{i=2}^{n} a_i m_i &= - \sum_{i=1}^{n} (a_i, m_i, e) = - \sum_{i=1}^{n} (e, a_i, m_i) = \\ = - \sum_{i=1}^{n} (e a_i) m_i &+ \sum_{i=1}^{n} e (a_i m_i) = - \sum_{i=1}^{n} (e a_i)(m_i e) = \\ = - e &\left(\sum_{i=1}^{n} a_i m_i \right) e = 0, \end{aligned}$$

proving the assertion.

Proposition 20. *Let A be an alternative MPR–ring with Baer radical $\beta(A)$.*

1) If $e \in A$ and $e^2 - e \in \beta(A)$, then there exists an element $v \in A$ such that $v^2 = v$ and $v - e \in \beta(A)$.

2) For any finitely many elements $a_1, \dots, a_n \in A$ there exists an idempotent $e \in A$ such that $a_i e - a_i \in \beta(A), e a_i - a_i \in \beta(A)$, for $i = 1, \dots, n$.

3) If I is a prime ideal of A, then A/I is isomorphic to a direct summand of $A/\beta(A)$.

PROOF. 1) Let B denote the subring of A generated by e. The ring B is clearly associative, and $B \cap \beta(A)$ is a nil–ideal of B. Since $e^2 - e \in \beta(A)$, the element $e^2 - e$ is idempotent modulo $B \cap \beta(A)$. Thus there exists an idempotent $v \in B$ such that $v - e \in B \cap \beta(A)$, and so $v - e \in \beta(A)$.

2) By Theorem 3 $A/\beta(A)$ is a direct sum of associative simple MPR–rings and of Cayley–Dickson algebras, and so for $a_1, \dots, a_n \in A$ there

exists an element e such that $e^2 - e,\ ea_i - a_i,\ a_ie - a_i \in \beta(A)$ for every $i = 1, \dots, n$. Thus by 1) there exists an idempotent $v \in A$ such that $v - e \in \beta(A)$. Clearly

$$va_i - a_i = (v - e)a_i + ea_i - a_i \in \beta(A)$$

holds, and also $a_iv - a_i \in \beta(A)$ is valid for every $i = 1, \dots, n$.

3) The assertion is an immediate consequence of Theorem 3.

In proving the next Proposition we shall make use of the representation theory of alternative algebras as given in [25] Chapter 11. If M and N are right ideals of an alternative ring A, such that $N \subseteq M$, then the canonical action of A on the factor group M/N defines an *alternative right A – module* (cf. [25] Proposition 11.1.4).

Proposition 21. *Let A be an alternative MPR–ring, M and N right ideals of A such that $N \subseteq M$. Let us suppose that for every right ideal L of A with $N \subseteq L \subseteq M$ either $L = N$ or $L = M$. Then*

1) M/N is an indecomposable alternative right A –module,

2) $M\mathcal{J}(A) \subseteq N$ where $\mathcal{J}(A)$ is the quasi–regular radical of A,

3) if $I = \{a \in A \mid Ma \subseteq N\}$, then $I \triangleleft A$ and A/I is either a Cayley–Dickson algebra or a simple associative ring with a minimal right ideal,

4) for every finitely many elements $m_1, \dots, m_n \in M$ there exists an idempotent $e \in A$ such that $m_ie - m_i \in N, i = 1, \dots, n$.

Proof. 1) By [25] Proposition 11.1.4 it follows that the canonical mapping $\rho : A \to \mathrm{End}_Z(M/N)$ is an alternative right representation of the ring A. Hence the assertion follows directly from the definitions (see [25] Chapter 11).

2) The statement follows from [25] Theorems 10.4.5 and 11.3.4.

3) By [25] Corollary on p23 q it follows that $I \triangleleft A$. Moreover, [25] Lemma 11.3.7 yields that A/I is a prime ring. Hence the statement follows from our Proposition 20.3.

4) By [25] Theorem 11.3.2 there exists an element $u \in M$ such that $u + N$ generates M/N and either

$$(ua)b + N = u(ab) + N \tag{3}$$

or

$$(ua)b + N = u(ba) + N \tag{4}$$

for all $a, b \in A$. Hence we conclude $M = uA + N$, and so there exist elements $a_1, \dots, a_n \in A$ such that

$$m_i - ua_i \in N, \qquad i = 1, \dots, n.$$

By Proposition 20.2 there exists an idempotent $e \in A$ such that

$$a_ie - a_i,\ ea_i - a_i \in \beta(A) \qquad i = 1, \ldots, n.$$

Hence using the identities (3) and (4) we have

$$m_ie + N = (ua_i)e + N = \begin{Bmatrix} u(a_ie) + N \\ u(ea_i) + N \end{Bmatrix} = ua_i + u\beta(A) + N.$$

Applying statement 2 of this Proposition, it follows

$$m_ie + N = ua_i + N = m_i + N, \qquad i = 1, \ldots, n.$$

Thus $m_ie - m_i \in N$ holds for $i = 1, \ldots, n$.

Let us recall that the *Loewy series* of a right ideal M of a ring A is defined as

$$\mathcal{L}_0(M) = \mathrm{Soc}\ M,$$
$$\mathcal{L}_{\alpha+1}(M)/\mathcal{L}_\alpha(M) = \mathrm{Soc}\ (M/\mathcal{L}_\alpha(M))$$

and

$$\mathcal{L}_\gamma(M) = \bigcup_{\alpha<\gamma} \mathcal{L}_\alpha(M)$$

for limit ordinals γ.

Proposition 22. *Let A be an alternative subdirectly irreducible MPR–ring with heart H. If H is a torsion ring, then so is A as well.*

PROOF. We prove the Proposition in four steps.

1) If $\beta(A) = 0$, then the assertion is just that of [25] Theorem 8.3.12.

2) Let us suppose that $\beta(A) \neq 0$. In this case also $\mathcal{J}(A) \neq 0$. Since the heart H is a torsion ring, there exists a prime p such that $pH = 0$. Consequently also the maximal torsion ideal T of A is a p–torsion ring.

We claim that A has no nonzero torsionfree right ideal. Let $M \neq 0$ be a torsionfree right ideal of A and N be a minimal principal right ideal of A such that $N \subseteq M$. Clearly also $N \cap T = 0$ holds, and by Proposition 14 also $N \subseteq D$ where D is the maximal divisible ideal of A. By Proposition 17 we have $AN + N \triangleleft A$, and so $H \subseteq AN + N$. Applying Proposition 21.4 and 19 for 0 and N, it follows that $(AN + N) \cap T = 0$, contradicting $0 \neq H \subseteq (AN + N) \cap T$. Thus $M \cap T \neq 0$ for every right ideal $M \neq 0$ of A.

3) Let M be a principal right ideal of A minimal with verpect to the properties $M \subseteq D$ and $M \not\subseteq T$. The right ideals $N = M \cap T$ and M satisfy clearly the conditions of Propositions 21, and hence $M\mathcal{J}(A) \subseteq N$ is valid.

We claim that $MJ = 0$ for $J = \mathcal{J}(A)$. We prove this by induction on α of the Loewy series $\mathcal{L}_\alpha(J)$.

By definition $\mathcal{L}_0(J) = \operatorname{Soc} J$. By step 2 we conclude $\operatorname{Soc} J \subseteq T$ and by $M \subseteq D$ Propositions 10 and 11 yield $M\mathcal{L}_0(J) = 0$.

Suppose that $M\mathcal{L}_\gamma(J) = 0$ for every $\gamma < \alpha$. If α is a limit ordinal, then $M\mathcal{L}_\alpha(J) = 0$. Let us consider the case when α is not a limit ordinal, and choose an arbitrary element $a \in \mathcal{L}_\alpha(J) \setminus \mathcal{L}_{\alpha-1}(J)$. Now

$$a + \mathcal{L}_{\alpha-1}(J) \in \operatorname{Soc}(J/\mathcal{L}_{\alpha-1}(J)),$$

and therefore there exist elements $a_0, a_1, \ldots, a_n \in J$ such that

$$(a) \qquad a + \mathcal{L}_{\alpha-1}(J) = \sum_{i=0}^{n} a_i + \mathcal{L}_{\alpha-1}(J),$$

(b) $qa_0 \in \mathcal{L}_{\alpha-1}(J)$ for some natural number q,

(c) every element $a_i + \mathcal{L}_{\alpha-1}(J)$ is contained in an indecomposable submodule of $\mathcal{L}_\alpha(J)/\mathcal{L}_{\alpha-1}(J)$.

By Proposition 21.4 there exist idempotents $e_1, \ldots, e_n \in A$ such that $a_ie_i - a_i \in \mathcal{L}_{\alpha-1}(J)$, $i = 1, \ldots, n$. Now we have

$$Ma \subseteq \sum_{i=0}^{n} Ma_i$$

and by the minimality of M also $qM = M$ holds. Hence by (b) and the hypothesis it follows

$$Ma_0 = (qM)a_0 = M(qa_0) \subseteq M\mathcal{L}_{\alpha-1}(J) = 0.$$

Let $m \in M$ be an arbitrary element. Then for every $i = 1, \ldots, n$ we have

$$ma_i = m(a_ie_i) - m(a_ie_i - a_i) = m(a_ie_i),$$

because

$$m(a_ie_i - a_i) \in m\mathcal{L}_{\alpha-1}(J) = 0.$$

Since $a_i \in J$, it follows $m(a_ie_i) \in mJ \subseteq N = M \cap T$. If $b_i = a_ie_i$, then

$$b_ie_i = (a_ie_i)e_i = a_i(e_i^2) = a_ie_i = b_i.$$

Hence from Proposition 18 we conclude that $mb_i = 0$ for every $i = 1, \ldots, n$. Thus $Ma_i = 0$ holds for every $i = 1, \ldots, n$. This together with $Ma_0 = 0$

imply $Ma = 0$, and so $M\mathcal{L}_\alpha(J) = 0$ is valid. Since A is an MPR–ring, $J = \mathcal{L}_\gamma(J)$ for an ordinal γ and by induction we get $MJ = 0$.

4) Let us consider elements $a_1, \dots, a_n \in A$ and $m_1, \dots, m_n \in M$, and suppose that $\sum_{i=1}^{n} m_i a_i \in T$. By Corollary 2 and Proposition 20.2 there exist an idempotent $e \in A$ such that $a_i e - a_i \in J$ for $i = 1, \dots, n$. Since $MJ = 0$, it follows that $m_i a_i = m_i(a_i e)$ for every $i = 1, \dots, n$. For the element $b_i = a_i e$ clearly

$$b_i e = (a_i e)e = a_i e^2 = a_i e = b_i, \qquad i = 1, \dots, n$$

holds. Hence we have

$$\sum_{i=1}^{n} m_i a_i = \sum_{i=1}^{n} m_i b_i.$$

In view of Proposition 14 for the MPR–ring A Proposition 18 is applicable with $m_1, \dots, m_n \in D$ and $b_1, \dots, b_n \in A$ and $\sum_{i=1}^{n} m_i b_i \in T$. Hence we obtain that $\sum_{i=1}^{n} m_i b_i = 0$, and therefore $MA \cap T = 0$. Since MA is a right ideal of A, by step 2 of this proof we conclude $MA = 0$. Hence by $M \subseteq D$, M_+ has d.c.c. on subgroups and so $M \subseteq T$. This contradicts $M \not\subseteq T$, consequently $D \subseteq T$ and A is a torsion ring, in fact a p–torsion ring.

Theorem 7. *Every alternative MPR–ring splits with respect to the torsion radical.*

PROOF. The statement is an immediate consequence of Theorem 6 and Proposition 22.

Remark. For associative MPR–rings the splitting of the maximal torsion ideal was proved by CHRISTINE AYOUB [1] and DINH VAN HUYNH [7]. In [23] Widiger proved the splitting of alternative artinian rings. Thus Theorem 7 generalizes both of these results.

6. Jordan rings and the torsion radical

In this section we shall consider *Jordan rings*, that is, rings which satisfy the identities

$$xy = yx$$
$$(x^2y)x = x^2(yx)$$

for all elements x and y. The second identity can be expressed also in the form

$$(x^2, y, x) = 0,$$

and by linearization one gets the identity

$$(xy, z, t) + (xt, z, y) + (yt, z, x) = 0 \tag{5}$$

for all elements x, y, z and t (cf. [25] Chapter 14, identity (22)).

Proposition 23. *Let A be a Jordan ring such that $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion subgroup and D is the maximal divisible ideal of A. Further, let T denote the maximal torsion ideal of A. If for any finitely many elements $x_1, \dots, x_n \in A/T$ there exists an element $e \in A/T$ such that $x_ie = x_i = x_ie^2$, $i = 1, \dots, n$, then D^2 is torsionfree.*

Proof. Substituting $x = y = e, z = a$ and $t = b$ into the identity (5), we get

$$(e^2, a, b) = -2(eb, a, e). \tag{6}$$

Let us assume that $\sum_{i=1}^{n} a_ib_i \in T$ for some elements $a_i, b_i \in D$; $i = 1, \dots, n$. Since by Proposition 11.4 it follows that there exists an element $e \in D$ such that

$$\begin{aligned} a_ie^2 &= a_i + r_i \\ a_ie &= a_i + s_i \\ b_ie &= b_i + t_i \end{aligned}$$

wher $r_i, s_i, t_i \in T$ for $i = 1, \dots, n$. Thus by $TD = DT = 0$ we have

$$\sum_{i=1}^{n} a_ib_i = \sum_{i=1}^{n}(a_i + r_i)b_i = \sum_{i=1}^{n}(e^2a_i)b_i = \sum_{i=1}^{n}(e^2(a_ib_i) + (e^2, a_i, b_i)) =$$

$$= e^2\sum_{i=1}^{n} a_ib_i + \sum_{i=1}^{n}(e^2, a_i, b_i) = \sum_{i=1}^{n}(e^2, a_i, b_i).$$

Applying (6) we get

$$\sum_{i=1}^{n} a_ib_i = -2\sum_{i=1}^{n}(eb_i, a_i, e) = -2\sum_{i=1}^{n}(((eb_i)a_i)e - (eb)(a_ie)) =$$

$$= -2\sum_{i=1}^{n}(((b_i+t_i)a_i)e-(b_i+t_i)(a_ie)) = -2\sum_{i=1}^{n}((b_ia_i)e-b_i(a_ie)) =$$

$$= -2\left(\sum_{i=1}^{n}a_ib_i\right)e+2\sum_{i=1}^{n}b_i(a_i+s_i) = 2\sum_{i=1}^{n}a_ib_i.$$

Hence

$$\sum_{i=1}^{n}a_ib_i = 2\sum_{i=1}^{n}a_ib_i$$

holds implying

$$\sum_{i=1}^{n}a_ib_i = 0,$$

and consequently D^2 is a torsionfree.

Let $T(A)$ denote the maximal torsion ideal of the Jordan ring A, and let $\mathcal{M}$ be the class of all Jordan rings such that

i) $A_+ = B_+ \oplus D_+$ where B_+ is a reduced torsion subgroup and D_+ the maximal divisible subgroup of A_+,

ii) for any finitely many elements $x_1, \dots, x_n \in A/T(A)$ there exists an element $e \in A/T(A)$ with $x_ie = x_i = x_ie^2$.

Clearly, condition i) is preserved under taking homomorphic images. Let I be any ideal of a ring $A \in \mathcal{M}$. Now we have $(T(A)+I)/I \subseteq T(A/I)$, and therefore there is a homomorphism φ as given below:

$$\begin{array}{ccc} A/T(A) & \xrightarrow{\varphi} & \dfrac{A/I}{T(A/I)} \\ \downarrow & & \uparrow \\ A/(T(A)+I) & \simeq & \dfrac{A/I}{(T(A)+I)/I} \end{array}$$

Let $\overline{x}_1, \dots, \overline{x}_n \in \frac{A/I}{T(A/I)}$ be arbitrary finitely many elements, and $x_1, \dots, x_n \in A/T(A)$ be elements such that $\varphi(x_i) = \overline{x}_i, i = 1, 2, \dots, n$. Since $A \in \mathcal{M}$, there exists an element $e \in A/T(A)$ such that $x_ie = x_i = x_ie^2$, and so for $\overline{e} = \varphi(e)$ we have $\overline{x}_i\overline{e} = \overline{x}_i = \overline{x}_i\overline{e}^2$. Thus also the factor ring A/I satisfies condition ii), proving that the class $\mathcal{M}$ is homomorphically closed. Hence Theorem 5 is applicable, and in view of Proposition 23 we arrive at

Corollary 3. *If a Jordan ring A satisfies the requirements of Proposition 23, then the maximal torsion ideal of A splits off.*

Acknowledgement. The paper was written when the second author visited the University of Florida, Gainessville in 1986 upon the invitation of Professor J. MARTINEZ and the University of Düsseldorf in 1987 under the auspices of the Alexander von Humboldt Foundation and when the first author visited the Matematical Institute of the Hungarian Academy of Sciences in 1990 under the financial support of Hungarian National Foundation for Scientific Research grant no. 1813. Sincere thanks are due to all of them.

References

[1] CHR. W. AYOUB, Conditions for a ring to be fissile, *Acta Math. Acad. Sci. Hungar.* **30** (1977), 233–237.

[2] R. L. BERNHARDT, Splitting hereditary torsion over semiperfect rings, *Proc. Amer. Math. Soc.* **22** (1969), 681–687.

[3] L. BICAN , T. KEPKA and P. NĚMEC, Rings modules and preradicals, *M. Dekker, New York*, 1982.

[4] B. BOSBACH, Komplementäre Halbgruppen, *Fund. Math.* **64** (1969), 257–287.

[5] B. BOSBACH, Zur Struktur der inversen Teilbarkeitshalbgruppen, *Semigroup Forum* **12** (1976), 137–143.

[6] A. H. CLIFFORD and G. B. PRESTON, The algebraic theory of semigroups, I., *AMS Providence*, 1961.

[7] DINH VAN HUYNH, Die Spaltbarkeit von MHR–Ringen, *Bull. Acad. Polon. Sci.* **25** (1977), 939–941.

[8] DINH VAN HUYNH and NGUYEN VIET DUNG, On the cardinality of ideals in artinian rings, *Arch. Math.* **49** (1988), 213–216.

[9] N. DIVINSKY , J. KREMPA and A. SULIŃSKI, Strong radical properties of alternative and associative rings, *J. Algebra* **17** (1971), 369–388.

[10] Z. FIEDOROWITZ, The structure of autodistributive algebras, *J. Algebra* **31** (1974), 427–436.

[11] B. J. GARDNER, Multiple radical theories, *Coll. Math.* **41** (1979), 345–351.

[12] J. S. GOLAN, Torsion theories, *Longman Sci. & Tech.*, 1986.

[13] K. R. GOODEARL, Singular torsion and the splitting properties, *AMS Memoirs* **124** (1972).

[14] T. KEPKA, On a class of non–associative rings, *Comm. Math. Univ. Carol.* **18** (1977), 531–540.

[15] A. KERTÉSZ, Lectures on artinian rings, *Akadémiai Kiadó, Budapest*, 1987.

[16] J. MARTINEZ, Algebra with weak distributivity, *Arch. Math.* **45** (1985), 133–143.

[17] B. I. PLOTKIN, On functorials, radicals and coradicals in groups (in Russian), *Mat. Zapiski* **3** (1970), 150–182.

[18] M. SLATER, Ideals in semiprime alternative rings, *J. Algebra* **8** (1968), 60–76.

[19] M. SLATER, Alternative rings with d.c.c., I., *J. Algebra* **11** (1969), 102–110.

[20] M. SLATER, The socle of an alternative ring, *J. Algebra* **14** (1970), 443–463.

[21] F. SZÁSZ, Über artinsche Ringe, *Bull. Acad. Polon. Sci.* **11** (1963), 351–354.

[22] M. V. VOLKOV, Separation of the radical in ring varieties, *Acta Sci. Math. (Szeged)* **46** (1983), 73–75.

[23] A. WIDIGER, Decomposition of artinian alternative rings, *Beiträge Alg. Geom., (Halle, DDR)* **12** (1982), 57–71.

[24] K. A. ZHEVLAKOV, Alternative artinian rings, *Alg. i Logika* (1966), 11–35.

[25] K. A. ZHEVLAKOV, A. M. SLIŃKO, I. P. SHESTAKOV and A. I. SHIRSHOV, Rings that are nearly associative, *Academic Press*, 1982.

K. I. BEĬDAR
SSSR 127562
MOSCOW
UL. SANNIKOVA
DOM 3, CORPUS 1, KV. 298

R. WIEGANDT
MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMYY OF SCIENCES
P.O. BOX 127
H-1364 BUDAPEST

(Received May 9, 1988; revised May 24, 1990)