

On the solutions of the functional equation

$$f(xf(y)^\ell + yf(x)^k) = tf(x)f(y)$$

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Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ denote sets of all positive integers, integers, rational numbers, and real numbers, respectively. The functional equation

$$(1) \quad f(xf(y)^\ell + yf(x)^k) = tf(x)f(y),$$

where $t \neq 0$ is a fixed real number, k and ℓ are fixed positive integers, and the unknown function f maps $\mathbf{R}$ into itself, has been studied by many authors in various cases (cf. [1] – [11]). N. BRILLOUËT [2] (cf. also [3], [4]) found all continuous solutions $f : \mathbf{R} \rightarrow \mathbf{R}$ of (1) in the case $k = \ell = 1$ and $t > 0$, M. SABLIK and P. URBAN [9] – [11] solved (1) for $t = 1$ in the class of continuous functions $f : \mathbf{R} \rightarrow \mathbf{R}$, and W. BENZ [1] determined the cardinality of the set of discontinuous solutions of (1) for many $t \in \mathbf{R}$.

We are going to find all continuous solutions of (1) in the case where $t \in \mathbf{R} \setminus \{0\}$ and $k \neq \ell$ are positive integers. We always suppose, without loss of generality, that $\ell < k$. The result presented here is a generalization of the one from [10].

Let us start with the following

Lemma 1. *If a function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies functional equation (1), then the following conditions hold:*

(i) *if there exists an $x \in \mathbf{R}$ such that $f(x) = 0$, then $f(0) = 0$,*

(ii) *if an $x_0 \in \mathbf{R} \setminus f^{-1}(\{0\})$, then the functions*

$$(2) \quad \mathbf{R} \ni y \rightarrow x_0 f(y)^\ell + y f(x_0)^k,$$

$$(3) \quad \mathbf{R} \ni y \rightarrow y f(x_0)^\ell + x_0 f(y)^k$$

are one – to – one.

PROOF. ad (i) Let $f(z) = 0$ and $x = y = z$. Then by (1) $f(0) = 0$.

ad (ii) We are going to prove that function (2) is one – to – one. The proof for function (3) is analogous.

Let us suppose that $f(x_0) \neq 0$ and function (2) is not one-to-one. Then there exist $y, z \in \mathbf{R}$ such that $y \neq z$ and

$$x_0 f(y)^\ell + y f(x_0)^k = x_0 f(z)^\ell + z f(x_0)^k.$$

Since by (1) $t f(x_0) f(y) = t f(x_0) f(z)$, so $f(y) = f(z)$ and $y f(x_0)^k = z f(x_0)^k$. Thus $y = z$. It is a contradiction.

This completes the proof.

Lemma 2. *If a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies (1) and there exists an $x \in \mathbf{R}$ such that $f(x) \neq 0$, then for each $r > 0$ there exists an $x_r \in \mathbf{R}$ such that $f(x_r) \neq 0$ and*

$$\left| \frac{f(x_r)^\ell}{x_r} \right| < r.$$

PROOF. Let us denote $F := \{x : f(x) \neq 0\}$, $s := \sup F$, and $i := \inf F$. Since f is continuous and there exists an $x \in \mathbf{R}$ such that $f(x) \neq 0$, so $F \neq \emptyset$ and $s \neq 0$ or $i \neq 0$.

The following two cases are possible:

1⁰ F is bounded,

2⁰ F is not bounded.

Suppose that 1⁰ holds. If $s \neq 0$, then there exists a sequence $\{x_n\}_{n \in \mathbf{N}} \subset F \setminus \{0\}$ such that $\lim_{n \rightarrow \infty} x_n = s$. If $s = 0$, then $i \neq 0$ and there exists a sequence $\{y_n\}_{n \in \mathbf{N}} \subset F \setminus \{0\}$ such that $\lim_{n \rightarrow \infty} y_n = i$.

Let $s \neq 0$. Then by 1⁰ $s < +\infty$ and $\lim_{n \rightarrow \infty} f(x_n) = 0$. Thus

$$\lim_{n \rightarrow \infty} \frac{f(x_n)^\ell}{x_n} = 0.$$

When $s = 0$, then in a similar way, we obtain that

$$\lim_{n \rightarrow \infty} \frac{f(y_n)^\ell}{y_n} = 0.$$

Now suppose that 2⁰ holds. Then there exists a sequence $\{x_n\}_{n \in \mathbf{N}} \subset F \setminus \{0\}$ such that $\lim_{n \rightarrow \infty} |x_n| = +\infty$. If there exists a subsequence $\{x_{n_k}\}_{k \in \mathbf{N}}$ of $\{x_n\}_{n \in \mathbf{N}}$ such that

$$(4) \quad \lim_{n \rightarrow \infty} \frac{f(x_{n_k})^\ell}{x_{n_k}} = 0,$$

then the assertion holds.

On the contrary, if each subsequence of $\{x_n\}_{n \in \mathbf{N}}$ does not satisfy (4), then there exist a $d > 0$ and an $n_0 \in \mathbf{N}$ such that

$$\left| \frac{f(x_n)^\ell}{x_n} \right| > d \quad \text{for all } n > n_0.$$

Hence

$$(5) \quad |f(x_n)^\ell| > d|x_n| \quad \text{for all } n > n_0$$

and

$$(6) \quad |x_n f(x_n)^{-\ell}| < \frac{1}{d} \quad \text{for all } n > n_0.$$

Since $\ell < k$ and $\lim_{n \rightarrow \infty} |x_n| = +\infty$, by (5) we have that

$$(7) \quad \lim_{n \rightarrow \infty} |f(x_n)^{k-\ell}| = +\infty.$$

Let

$$W(x, y) := x f(y)^\ell + y f(x)^k \quad \text{for all } x, y \in \mathbf{R}.$$

Then

$$\frac{f(W(x, y))^\ell}{W(x, y)} = \frac{t^\ell f(x)^\ell f(y)^\ell}{x f(y)^\ell + y f(x)^k} = \frac{t^\ell f(y)^\ell}{x f(x)^{-\ell} f(y)^\ell + y f(x)^{k-\ell}} \quad \text{for all } x, y \in F \setminus \{0\}.$$

Thus by (6) and (7)

$$\lim_{n \rightarrow \infty} \frac{f(W(x_n, y))^\ell}{W(x_n, y)} = 0 \quad \text{for all } y \in F \setminus \{0\}.$$

This implies the assertion.

Lemma 3. *If a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies (1) and there exist $a, b \in \mathbf{R}$, $a < b$ such that $f(a) = f(b) \neq 0$, then $f(x) = f(a)$ for all $x \in [a, b]$.*

PROOF. Let us denote

$$V(x, y, z) := \frac{f(x)^\ell}{x} (y - z) + f(y)^k - f(z)^k \quad \text{for all } x, y, z \in \mathbf{R}, x \neq 0$$

and suppose that there exists a $c \in (a, b)$ such that $f(c) \neq f(a)$. Then

$$(8) \quad \operatorname{sgn}(f(c)^k - f(a)^k) \neq \operatorname{sgn}(f(b)^k - f(c)^k),$$

where

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Since

$$|f(c)^k - f(a)^k| = |f(b)^k - f(c)^k| \neq 0,$$

by Lemma 2 there exists an $x_0 \in \mathbf{R}$ such that

$$\left| \frac{f(x_0)^\ell}{x_0} (c - a) \right| < |f(c)^k - f(a)^k|$$

and

$$\left| \frac{f(x_0)^\ell}{x_0} (b - c) \right| < |f(b)^k - f(c)^k|.$$

Hence

$$(9) \quad \operatorname{sgn} V(x_0, c, a) = \operatorname{sgn}(f(c)^k - f(a)^k)$$

and

$$(10) \quad \operatorname{sgn} V(x_0, b, c) = \operatorname{sgn}(f(b)^k - f(c)^k).$$

It is easy to observe that by (8), (9), and (10)

$$(11) \quad \operatorname{sgn} V(x_0, b, a) \neq \operatorname{sgn} V(x_0, c, a)$$

or

$$(12) \quad \operatorname{sgn} V(x_0, b, a) \neq \operatorname{sgn} V(x_0, b, c).$$

Since f is continuous, the following functions:

$$\mathbf{R} \ni y \rightarrow V(x_0, y, a),$$

$$\mathbf{R} \ni y \rightarrow V(x_0, b, y)$$

are continuous. Thus, if (11) holds, then there exists an $e_1 \in [c, b]$ such that

$$(13) \quad V(x_0, e_1, a) = 0$$

and if (12) holds, then there exists an $e_2 \in [a, c]$ such that

$$(14) \quad V(x_0, b, e_2) = 0.$$

It is easy to verify that (13) implies that

$$e_1 f(x_0)^\ell + x_0 f(e_1)^k = a f(x_0)^\ell + x_0 f(a)^k$$

and (14) implies that

$$e_2 f(x_0)^\ell + x_0 f(e_2)^k = b f(x_0)^\ell + x_0 f(b)^k.$$

In view of Lemma 1 (ii) it is impossible.

Lemma 4. *If a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies (1) and there exist $a, b \in \mathbf{R}, a < b$ such that $f(x) = f(a) = f(b) \neq 0$ for all $x \in (a, b)$, then $f(x) = f(a)$ for all $x \in \mathbf{R}$.*

PROOF. Let us suppose that there exists an $x \in \mathbf{R}$ such that $f(x) \neq f(a)$. Then $\text{card } f(\mathbf{R}) > \text{card } \mathbf{N}$ and by axiom of choice there exists a set $Y \subset \mathbf{R}$ such that

$$(15) \quad \text{card } Y > \text{card } \mathbf{N},$$

$$(16) \quad f(x) \neq f(y), \quad \text{for all } x, y \in Y, x \neq y,$$

and

$$(17) \quad 0 \notin f(Y).$$

Fix a $y \in \mathbf{R}$. We define a function $W^y : \mathbf{R} \rightarrow \mathbf{R}$ as follows

$$W^y(x) := y f(x)^\ell + x f(y)^k, \quad \text{for all } x \in \mathbf{R}.$$

It follows from (1) that $f(W^y(x)) = t f(x) f(y)$ for all $x, y \in \mathbf{R}$. Thus

$$f(z) = t f(a) f(y), \quad \text{for all } z \in W^y([a, b]), y \in \mathbf{R}$$

and according to (16) and (17)

$$(18) \quad W^x([a, b]) \cap W^y([a, b]) = \emptyset, \quad \text{for all } x, y \in Y, x \neq y.$$

It is easy to notice that for each $y \in Y$ the function W^y is continuous and by Lemma 1 (ii) is one - to - one. Hence $W^y([a, b])$ is a nontrivial interval for all $y \in Y$.

Let $A := \{W^y([a, b]) : y \in Y\}$. Then (18) implies that A is a family of disjoint intervals. By (15) $\text{card } A > \text{card } \mathbf{N}$. It is impossible.

Finally we prove the following

Theorem 1. *The functions $f_1 \equiv 0$ and $f_2 \equiv \frac{1}{t}$ are the only continuous solutions of (1) in the class of functions $f : \mathbf{R} \rightarrow \mathbf{R}$.*

PROOF. Let a continuous function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfy (1) and let $f \not\equiv 0$. We define $F := \{x : f(x) \neq 0\}$. Thus

$$(19) \quad F \neq \emptyset.$$

Two cases are possible

- (a) $f|_F$ is not one - to - one,
- (b) $f|_F$ is one - to - one,

where the function $f|_F : F \rightarrow \mathbf{R}$ is defined as follows

$$f|_F(x) := f(x) \quad \text{for all } x \in F.$$

When (a) holds, then there exist $a, b \in F, a < b$ such that $f(a) = f(b) \neq 0$. Thus by Lemma 3 we obtain that $f|_{[a,b]} \equiv f(a) \neq 0$ and consequently on account of Lemma 4 and (1) $f \equiv \frac{1}{t}$.

On the contrary, if (b) holds, then by (1) the following condition is true:

$$x f(y)^k + y f(x)^\ell = y f(x)^k + x f(y)^\ell, \quad \text{for all } x, y \in F.$$

Thus

$$f(x)^\ell - f(x)^k = x \frac{f(y)^\ell - f(y)^k}{y} \quad \text{for all } x, y \in F, y \neq 0.$$

Since f is continuous and $f \not\equiv \text{const}$, there exists a $y_0 \in F \setminus \{0\}$ such that $f(y_0)^\ell \neq f(y_0)^k$. Denote

$$p := \frac{f(y_0)^\ell - f(y_0)^k}{y_0}.$$

Then

$$(20) \quad f(x)^\ell - f(x)^k = px, \quad \text{for all } x \in F.$$

Hence the function $g : \mathbf{R} \rightarrow \mathbf{R}$ defined as follows

$$(21) \quad g(x) = \frac{1}{p}(x^\ell - x^k), \quad \text{for all } x \in \mathbf{R}$$

satisfies the following condition

$$(22) \quad g(f(F)) = F.$$

Now suppose that

$$(23) \quad F = \mathbf{R}.$$

Then by the fact that f is continuous we obtain that $f(F)$ is connected. Thus $f(F) \subset (0, +\infty)$ or $f(F) \subset (-\infty, 0)$. Since by (21) $g((0, +\infty)) \neq \mathbf{R}$ and $g((-\infty, 0)) \neq \mathbf{R}$, by (22) we obtain that (23) does not hold. Hence by Lemma 1 (i)

$$(24) \quad f(0) = 0.$$

It follows from (21) that $g(1) = g(0) = 0$. Thus by (24) and (22)

$$(25) \quad \{0, 1\} \cap f(F) = \emptyset.$$

Observe that since f is continuous and (b) holds, we obtain that $f(F) \cup \{0\} = f(\mathbf{R})$ is connected and $f^{-1}(\{0\})$ is connected. Thus by (19) F is unbounded and consequently according to (20) $f(F)$ is unbounded. Therefore in view of (25)

$$(26) \quad f(F) = (-\infty, 0).$$

Hence and from (21) and (22) it follows that

$$(27) \quad \text{card } \mathbf{R} \setminus F > 2.$$

In the case where k and ℓ are even numbers, we have that $g(-1) = 0$. Thus by (22), (24), and (26) at least one of the following cases holds:

(j) k is an odd number,

(jj) ℓ is an odd number.

Suppose that (j) holds and denote $A_k := \{f(x)^k : x \in F\}$. Then on account of (26)

$$(28) \quad A_k = (-\infty, 0).$$

Since according to (1)

$$f(x f(y)^k) = 0, \quad \text{for all } y \in F, x \in f^{-1}(\{0\}),$$

so

$$(29) \quad x A_k := \{x z : z \in A_k\} \subset f^{-1}(\{0\}), \quad \text{for all } x \in f^{-1}(\{0\})$$

and

$$(30) \quad z f^{-1}(\{0\}) \subset f^{-1}(\{0\}), \quad \text{for all } z \in A_k.$$

Fix any $x_0 \in f^{-1}(\{0\}) \setminus \{0\}$ (cf. (27)). If $x_0 < 0$, then by (28) and (29)

$$x_0 A_k = (0, +\infty) \subset f^{-1}(\{0\})$$

and consequently in virtue of (28) and (30)

$$(-\infty, 0) \subset f^{-1}(\{0\}).$$

Thus according to (24)

$$f^{-1}(\{0\}) = \mathbf{R}.$$

On the contrary, if $x_0 > 0$, then by (28) and (29)

$$x_0 A_k = (-\infty, 0) \subset f^{-1}(\{0\})$$

and consequently in view of (28) and (30)

$$(0, +\infty) \subset f^{-1}(\{0\}).$$

Thus on account of (24)

$$f^{-1}(\{0\}) = \mathbf{R}.$$

Hence $f \equiv 0$. It leads to a contradiction.

In the case (jj) we obtain a contradiction in the same way.

This completes the proof.

We can generalize the above theorem in a similar way as in [10].

Theorem 2. Let X be a linear space over $\mathbf{R}$ and let a function $f : X \rightarrow \mathbf{R}$ satisfy (1). Then, if for each $x \in X \setminus \{0\}$ the function $g_x : \mathbf{R} \rightarrow \mathbf{R}$ given by

$$g_x(s) = f(sx) \quad \text{for all } s \in \mathbf{R}$$

is continuous, then $f \equiv 0$ or $f \equiv \frac{1}{t}$.

PROOF. It is easy to verify that g_x satisfies (1) for $x \in X \setminus \{0\}$. Thus by Theorem 1 $g_x \equiv \text{const}$ for $x \in X \setminus \{0\}$.

Since $g_x(0) = f(0)$ for $x \in X \setminus \{0\}$, so $g_x \equiv f(0)$ for $x \in X \setminus \{0\}$ and therefore $f \equiv \text{const}$. This completes the proof.

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References

- [1] W. BENZ, The cardinality of the set of discontinuous solutions of a class of functional equations, *Aequationes Math.* **32** (1987), 58–62.
- [2] N. BRILLOUËT, Equations fonctionnelles et recherche de sous – groups, *Publ. Math. Univ. Nantes*, (1983).
- [3] N. BRILLOUËT, J. DHOMBRES, Equations fonctionnelles et recherche de sous – groups, *Aequationes Math.* **31** (1986), 253–293.
- [4] J. DHOMBRES, Finding subgroups, *Aequationes Math.* **24** (1982), 267–269.
- [5] S. MIDURA, Sur la determination de certains sous – groups du groupe L_3^1 à l'aide d'équations fonctionnelles, *Dissertationes Math.* **105** (1973).
- [6] S. MIDURA, Sur certains sous – demigroupes à un parametre du groupe L_3^1 déterminés à l'aide d'équations fonctionnelles, Proc. of the Twenty – third Internat. Symposium on Functional Equations, Gargnano, Italy, June 2 – June 11, 1985, *Centre for Information Theory*, 26–27, *Faculty of Mathematics, Waterloo, Ontario, Canada*.
- [7] S. MIDURA, Sur certains sous – demigroupes à un parametre des groupes L_2^1 et L_3^1 déterminés à l'aide d'équations fonctionnelles, *Rocz. Nauk. Dyd. WSP Rzeszów* **7/62** (1985), 37–50.
- [8] S. MIDURA, Solutions des équations fonctionnelles qui determinent les sous – demigroupes du groupe L_4^1 , *Rocz. Nauk. Dyd. WSP Rzeszów* **7/62** (1985), 51–56.
- [9] M. SABLIK, Remark, *Aequationes Math.* **26** (1984), 274.
- [10] M. SABLIK, P. URBAN, On the solutions of the equation $f(x f(y)^k + y f(x)^\ell) = f(x)f(y)$, *Demonstratio Math.* **18** (1985), 863–867.
- [11] P. URBAN, Continuous solutions of the functional equation $f(x f(y)^k + y f(x)^\ell) = f(x)f(y)$, *Demonstratio Math.* **16** (1983), 1019–1025.

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