Publ. Math. Debrecen
38 / 3–4 (1991), 215–223

On the approximate realizability of certain spaces of transfer functions

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In the present paper we consider an infinite dimensional realization problem. If the transfer function $H(s)$ is a fractional matrix then it can be realized by a finite dimensional system, described by an ordinary differential equation:

$$\dot{x} = Ax + Bu, \quad y = Cx.$$

Now, transfer functions belonging to the Hardy space H^2_R of analytic functions will be realized by partial differential equations. We shall prove that the functions $H \in H^2$, realizable by an infinite dimensional system described by a partial differential equation and an observation equation

$$\begin{aligned} \partial_t u + \partial_x u &= bv, \\ y &= < c, u >, \end{aligned} \tag{1}$$

form a dense subset in H^2. This means, that all functions $H \in H^2$ are approximately realizable by systems of type (1). The realizations are defined in terms of b, c belonging to a very naturally defined Banach space. We mention that the author in her thesis [4] has defined such a subspace X of H^2 equipped with a complete norm, that all functions belonging to X are realizable by (1) . Nevertheless, the author prefers the results and methods of this paper in spite of the fact that these results deal only with approximately realizability (see also [3]).

1. Preliminaries

Now we define a Banach space $X_R (R > 0)$ of measurable functions $g : \mathcal{R} \to \mathcal{R}(\mathcal{C})$ satisfying the following condition:

$$\|g\|_{X_R} = \sup_{i \in \mathcal{Z}_+} R^i \int_{\mathcal{R}} |g(x)| \quad \frac{|x|^i}{i!} dx < \infty.$$

The space X_R^+ is defined analogously:

$$X_R^+ = \left\{ g : \mathcal{R}_+ \to \mathcal{R} : \|g\|_{X_R^+} = \sup_{i \in \mathcal{Z}_+} R^i \int\limits_{\mathcal{R}_+} |g(x)\frac{|x|^i}{i!} dx < \infty \right\}.$$

Consider the family of operators

$$\Phi(t) : X_R \to X_R^+, \quad t \geq 0$$

defined by

$$(\Phi(t)g)(x) = g(x - t).$$

Lemma 1. *The inequality*

$$\|\Phi(t)\| \leq \exp tR$$

holds for each $t \geq 0$.

Indeed, for each $i \in \mathcal{Z}_+,\ g \in X_R$,

$$\|\Phi(t)g\|_{X_R^+} = \sup_{i \in \mathcal{Z}_+} R^i \int\limits_{\mathcal{R}_+} |g(x-t)|\frac{|x|^i}{i!} dx \leq$$

$$\leq \sup_{i \in \mathcal{Z}_+} R^i \sum_{j=0}^{i} \int\limits_{\mathcal{R}} |g(x)|\frac{|x|^{i-j}}{(i-j)!} dx \frac{t^j}{j!} \leq \|g\|_{X_R} \exp Rt$$

Now define the Banach space Y_R of strongly measurable functions $f : \mathcal{R}_+ \to X_R$ satisfying the condition

$$\|f\|_{Y_R} = \int\limits_{\mathcal{R}_+} \|f(s)\|_{X_R} \exp(-sR) ds < \infty.$$

Define the operators $\Psi(t) : Y_R \to X_R^+, \quad t \geq 0$, by

$$\Psi(t)f = \int\limits_0^t \Phi(t-s)f(s) ds.$$

Lemma 2. *The inequality*

$$\|\Psi(t)\| \leq \exp Rt$$

holds for each $t \geq 0$.

Indeed, for each $f \in Y_R$,

$$\|\Psi(t)f\|_{X_R^+} \leq \int_0^t \|f(s)\|_{X_R} \exp(t-s)Rds \ \leq \ \|f\|_{Y_R} \ \exp Rt.$$

Consider the functions $f \in Y_R, \quad g_0, g_1 \in X_R^+$.
Let $g \in X_R$ be defined by $g|_{\mathcal{R}_+} = g_0$ and $g(t) = g_1(-t) \quad (t < 0)$.
Then the generalized solution $u : \mathcal{R}_+ \to X_{\mathcal{R}}^+$ of the initial value problem

$$\begin{aligned} &\partial_t u + \partial_x u = f \\ &u(0,\cdot) = g_0, \quad u(\cdot, 0) = g_1 \end{aligned} \tag{2}$$

is given by the formula

$$(t,x) \mapsto u(t,x) = u(t)(x) = (\Phi(t)g)(x) + (\Psi(t)f)(x) =$$

$$= \begin{cases} g_0(x-t) + \int\limits_0^t f(s, x-t+s)ds & (x \geq t), \\ g_1(t-x) + \int\limits_{t-x}^t f(s, x-t+s)ds & (x < t). \end{cases}$$

Then, by Lemmas 1, 2, the inequality

$$\|u(t)\|_{X_R^+} \leq (\|f\|_{Y_R} + \|g\|_{X_R}) \exp \ tR \tag{3}$$

holds for each $t \geq 0$.

We mention that if f, g, are C^1-functions, $g_0, g_1 : \mathcal{R}_+ \to \mathcal{R}$ are also C^1-functions then u is a C^1-solution of (2) in the ordinary sense over $\mathcal{R}_+ \times \mathcal{R}_+ \setminus \{x = t\}$. This fact justifies that (3) will be called a generalized solution of (2) (see [1]).

From inequality (3) we get that

$$\begin{aligned} \sup_{t \in \mathcal{R}_+} \|u(t)\|_{X_R} \exp(-tR) &\leq \|f\|_{Y_R} + \|g\|_{X_R} = \\ &= \|f\|_{Y_R} + \|g_0\|_{X_R^+} + \|g_1\|_{X_R^+}. \end{aligned}$$

Now we turn to the definition of control functions. The space C_R consists of the functions $v : \mathcal{R}_+ \to \mathcal{R}$ satisfying the following condition

$$\|v\|_{C_R} = \int_{\mathcal{R}_+} |v(s)| \exp(-sR)ds \ < \infty.$$

If $v \in C_R$ then, for each $b \in X_R^+$, the function

$$(t,x) \mapsto f(t,x) = \begin{cases} v(t)b(x) & (t, x \geq 0), \\ 0 & (x < 0) \end{cases} \tag{4}$$

belongs to Y_R. Indeed, $f : \mathcal{R}_+ \to X_R$ is strongly measurable, $f(t,\cdot)\,|_{\mathcal{R}_-} = 0$ and

$$\|f\|_{Y_R} = \int_{\mathcal{R}_+} \|v(t)b\|_{X_R} \exp(-tR)dt = \|b\|_{X_R^+} \|v\|_{C_R}.$$

If we have a function $b \in X_R^+$, then the differential equation

$$\partial_t u + \partial_x u = bv \tag{5}$$

will be understood in sense (4) and (5) will be called a control system.

Now define the observation of this system. Let $C \in (X_R^+)^*$. Then

$$y = < c, u >$$

is the observation of the control system (5), where $< \cdot, \cdot >$ is the dual pair map of (X_R^+, X_R^{+*}).

Define the space O_R of the observation. O_R consists of the functions $y : \mathcal{R}_+ \to \mathcal{R}$, such that

$$\|y\|_{O_R} = \sup_{t \geq 0} \{|y(t)| \exp(-Rt)\} < \infty.$$

Lemma 3. *The input-output map defined by the system*

$$\begin{aligned} &\partial_t u + \partial_x u = bv, \quad \mu(0,\cdot) = g_0, \quad \mu(\cdot, 0) = g_1, \\ &y = < c, u > \end{aligned} \tag{6}$$

is a linear and continuous

$$X_R^+ \times X_R^+ \times C_R \to O_R$$

operator.

Indeed,

$$\begin{aligned}
\|y\|_{O_R} &= \sup_{t\geq 0}\{|y(t)|\exp(-tR)\} \leq \sup_{t\geq 0}\{|<c, u(t,\cdot)>|\exp(-tR)\} \leq \\
&\leq \|c\|_{X_R^{+*}} \sup_{t\geq 0}\{\|u(t,\cdot)\|_{X_R^+}\exp(-tR)\} \leq \\
&\leq \|c\|_{X_R^{+*}} \left(\|b\|_{X_R^+}\|v\|_{C_R} + \|g_0\|_{X_R^+} + \|g_1\|_{X_R^+}\right) \leq \\
&\leq \|c\|_{X_R^{+*}} \left(1 + \|b\|_{X_R^+}\right)\left(\|v\|_{C_R} + \|g_0\|_{X_R^+} + \|g_1\|_{X_R^+}\right)
\end{aligned}$$

We notice that Lemma 3 shows that if the control v grows exponentially then the corresponding observation also grows exponentially. Thus the Laplace transform of both functions will be convergent over the open half plane $\{Re\ z > R\}$. Thus the transfer function of the system (6) can be defined in terms of the Laplace transformed equations (6). To do so, suppose that the initial conditions g_0, g_1 are equal to O. Then

$$Ly = HLv$$

Lemma 4. *The transfer function H satisfies the equality*

$$H(s) = \int_{\mathcal{R}_+} \int_0^x \exp s(\xi - x) b(\xi) d\xi\ c(x) dx. \tag{7}$$

Indeed,

$$\begin{aligned}
s(Lu)(s) + \partial_x(Lu)(s) &= b(Lv)(s) \\
(Ly)(s) &= \int_{\mathcal{R}_+} c(Lu)(s).
\end{aligned}$$

From these equations we get that

$$(Ly)(s) = \int_{\mathcal{R}_+} \int_0^x \exp s(\xi - x)\ b(\xi) d\xi\ c(x) dx (Lv)(s)$$

that is, (7) holds.

We shall say that the system (5) is a realization of the function H if the equality (7) holds. It is obvious that if a function H has a realization of type (6), then H is analytic over the half plane $\{Re\, z > R\}$.

The following lemma justifies a generalization. Now we recall that, in the Hardy space of the analytic functions over the disk of radius R with square integrable trace over the boundary ∂D_R, the scalar product is defined by

$$< f,g >= \int_0^1 f(R\exp(2\pi i\varphi))g(R\exp(-2\pi i\varphi))d\varphi.$$

Lemma 5. *Let $\varepsilon > 0$, $b, c \in X^+_{R+\varepsilon}$. Then the function H defined by (7) belongs to H^2_R.*

PROOF. If a function $g \in X^+_{R+\varepsilon}$ then the function

$$s \mapsto \sum_{k=0}^{\infty} s^k \int_{\mathcal{R}_+} |g(x)| \frac{x^k}{k!} dx \tag{8}$$

belongs to the space H^2_R. If $g = 0$ then this statement is obvious. If $g \neq 0$ then

$$1 = \overline{\lim} \quad \|g\|^{1/k}_{X^+_{R+\varepsilon}} \geq (R+\varepsilon)\overline{\lim}\left(\int_{\mathcal{R}_+} |g(x)| \frac{x^k}{k!} dx\right)^{\frac{1}{k}},$$

thus the convergence radius r of (8) satisfies the inequality

$$r = \frac{1}{\overline{\lim}(\int\limits_{R_+} |g(x)| \frac{x^k}{k!})^{\frac{1}{k}}} \geq R + \varepsilon$$

Now we can estimate the series of the function $H(s)$.

$$|H(s)| \leq \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \int_{\mathcal{R}_+} \int_0^x \frac{\xi^i}{i!} |b(\xi)|\, d\xi \, \frac{x^{k-i}}{(k-i)!} |c(x)| \, dx \right) |s|^k \leq$$

$$\leq \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \int_{\mathcal{R}_+} \int_{\mathcal{R}_+} \frac{\xi^i}{i!} |b(\xi)| \, d\xi \, |s|^i \frac{x^{k-i}}{(k-i)!} |c(x)| \, dx \, |s|^{k-i} =$$

$$= \left(\sum_{i=0}^{\infty} \int_{\mathcal{R}_+} |b(\xi)| \frac{\xi^i}{i!} \, dx \, |s|^i \right) \left(\sum_{k=0}^{\infty} \int_{\mathcal{R}_+} \frac{x^k}{k!} |c(x)| \, dx \, |s|^k \right).$$

Thus, by the first part of this proof, the series of H is analytic over the disk $D_{R+\varepsilon}$ consequently H belongs to H^2_R.

We shall say that a function H is weakly realizable by the system (6) if there exist $b, c \in X_R^+$ such that (7) holds. It is not true all elements of the space are weakly realizable. Nevertheless, we can consider the approximate realizability of the space H^2_R. We say that a space X of analytic functions is approximately realizable if the subset of weakly realizable functions is dense in X.

2. The approximate realizability of H^2_R

In this section we prove a theorem on the approximate realizability of the space H^2_R, for each $R > 0$. Let $\varepsilon > 0$ and $b \in X^+_{R+\varepsilon}$. Define the operator $H_b : X^+_{R+\varepsilon} \to H^2_R$ by

$$(9) \qquad H_b(c)(s) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} (-1)^{k-i} \int\limits_{\mathcal{R}_+} \int\limits_0^x \frac{\xi^i}{i!} b(\xi)\, d\xi\, c(x) \frac{x^{k-i}}{(k-i)!} dx \right) s^k.$$

We shall prove that following lemma.

Lemma 6. *The operator $H_b : X^+_{R+\varepsilon} \to H^2_R$ is linear and continuous.*

Indeed, by Lemma 5, the operator H_b is well defined. The linearity is obvious. The continuity follows from the estimate

$$|(H_b c)(s)| \le \left(\sum_{i=0}^{\infty} \int\limits_{\mathcal{R}_+} |b(\xi)| \frac{\xi^i}{i!} d\xi R^i \right) \left(\sum_{k=0}^{\infty} \int\limits_{\mathcal{R}_+} |c(x)| \frac{x^k}{k!} dx R^k \right) \le$$

$$\le \left(\sum_{k=0}^{\infty} \left(\frac{R}{R+\varepsilon} \right)^k \right)^2 \|b\|_{X^+_{R+\varepsilon}} \cdot \|c\|_{X^+_{R+\varepsilon}} = \left(\frac{R+\varepsilon}{\varepsilon} \right)^2 \|b\|_{X^+_{R+\varepsilon}} \cdot \|c\|_{X^+_{R+\varepsilon}},$$

for each $|s| = R$. Thus $\|H_b c\|_{H^2_R} \le \left(\frac{R+\varepsilon}{\varepsilon}\right)^2 \|b\|_{X^+_{R+\varepsilon}} \|c\|_{X^+_{R+\varepsilon}}$, that is

$$\|H_b\| \le \left(\frac{R+\varepsilon}{\varepsilon} \right)^2 \|b\|_{X^+_{R+\varepsilon}}.$$

Now we are able to prove our main theorem.

Theorem. *Let $R > 0$. Then H_R^2 is approximately realizable by a system of type (6).*

PROOF. Let $\varepsilon > 0$, $b \in X_{R+\varepsilon}^+$. Suppose that $b \neq 0$. Then we define the operator $H_b : X_{R+\varepsilon} \to H_R^2$ by (9).

We recall the orthogonality relation

$$H_R^2 = \overline{R(H_b)} \oplus N(H_b^*).$$

Thus we need to prove that the kernel of the adjoint operator H_b^* is 0. By the definition of the adjoint and by the Riesz representation $H_R^{2^*} = H_R^2$ we have

$$\begin{aligned}
< H_b^* H, c >= &< H_b c, H >= \\
&= \int_0^1 \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} \int_{\mathcal{R}_+} \int_0^x \frac{\xi^j}{j!} b(\xi) d\xi \, c(\xi) \frac{x^{k-j}}{(k-j)!} dx \cdot \\
&\quad \cdot H\left(R \exp(2\pi i \varphi)\right) R^k \exp(-2k\pi i \varphi) d\varphi = \\
&= \int_{\mathcal{R}_+} \int_0^x \left(\sum_{k=0}^{\infty} \frac{(\xi - x)^k}{k!} \hat{H}(k) \right) b(\xi) d\xi \, c(x) dx = \\
&= \int_{\mathcal{R}_+} \int_0^x h(x - \xi) b(\xi) d\xi \, c(x) dx = 0
\end{aligned}$$

for each $c \in X_{R+\varepsilon}^+$ thus

$$x \longmapsto \int_0^x h(x - \xi) b(\xi) d\xi = 0, \tag{10}$$

where

$$\hat{H}(k) = \int_0^1 H(R \exp 2\pi i \varphi) R^k \exp(-2\pi k i \varphi) d\varphi, \quad k \in \mathcal{Z}_+,$$

are the Fourier coefficients of H in H_R^2 and the function h is defined by

$$\xi \longmapsto h(\xi) = \sum_{k=0}^{\infty} \frac{(-\xi)^k}{k!} \hat{H}(k).$$

By the Titchmarsh theorem, from (10), it follows that $h = 0$, that is $\hat{H}(k) = 0$ for each k. Thus the equality $H_b^* H = 0$ implies $H = 0$. So, by the orthogonality relation, it follows that, for each $b \neq 0$, the range $R(H_b)$ is dense in H_R^2.

Remark. Let $b \in X_{R+\varepsilon}^{+}$. Suppose that $b \neq 0$. Then, for each $H \in H_R^2$, there exists a sequence $(c_n) : \mathcal{N} \longrightarrow X_{R+\varepsilon}^{+}$ such that the transfer function H_n , corresponding to the system of type (6) described by $b, c_n \in X_{R+\varepsilon}^{+}$ converges to H on H_R^2. This means that the function b can also be prescribed and only the functions c_n depend on H.

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(Received December 18, 1987)