

Skew pure projective abelian groups

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In [2], we have investigated skew projective abelian groups. Recently J.D. REID and BERLINGHOFF have studied the structure of quasi-pure projective abelian groups. In this paper we study the structure of skew pure projective (s.p.p.) torsion abelian groups. We prove the following theorem assuming the generalized continuum hypothesis.

Theorem. A torsion abelian group $G = \sum G_p$ is skew pure projective if and only if each G_p is a direct sum of cyclic groups, or a direct sum of a divisible group and a bounded p -group (primary p -group).

All groups considered are abelian.

Section 1. We start with the following:

Definition 1.1. A group G is called skew pure projective if every diagram of the following type

$$\begin{array}{ccccccc} 0 & \rightarrow & S & \rightarrow & G & \xrightarrow{\eta} & G/S \rightarrow 0 \\ & & & & & \downarrow \alpha & \\ & & & & & G/S & \end{array}$$

with S pure in G can be completed to

$$\begin{array}{ccccccc} 0 & \rightarrow & S & \rightarrow & G & \xrightarrow{\eta} & G/S \rightarrow 0 \\ & & & & \downarrow \theta & & \downarrow \alpha \\ 0 & \rightarrow & S & \rightarrow & G & \xrightarrow[\eta]{} & G/S \rightarrow 0 \end{array}$$

where η is the natural map and α and θ are endomorphisms of G/S and G respectively such that $\alpha\eta = \eta\theta$.

Remark 1.2. Every direct summand of an s.p.p. group G is s.p.p.

The proof of the above remark is obvious.

Remark 1.3. A torsion group G is s.p.p. if and only if each of its primary components is.

PROOF. We notice that if $G = \sum G_p$ and S is a pure subgroup of G , and $S = \sum S_p$ then $G/S \simeq \sum G_p/S_p$ and since the G_p 's are fully invariant, the endomorphisms of G/S are specified by those of G_p/S_p .

Section 2. In view of remark 1.3, in this section we consider only p-primary s.p.p. groups and we study their structure.

Lemma 2.1. Let $G = A \oplus B$ be s.p.p. If A is a pure epimorphic image of B , then $A \simeq$ to a direct summand of B .

PROOF. Let $\beta : A \simeq B/S$ where S is a pure subgroup (p.s.g.) of B . Consider the following diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & S & \rightarrow & A \oplus B & \xrightarrow{A+\delta=\nu} & A \oplus B/S \rightarrow 0 \\ & & & & \downarrow \theta & & \downarrow \alpha \\ 0 & \rightarrow & S & \rightarrow & A \oplus B & \xrightarrow{A+\delta=\nu} & A \oplus B/S \rightarrow 0 \end{array}$$

Let α be the endomorphism of $A \oplus B/S$ where $\alpha|_{B/S} = \beta^{-1}$ and $\alpha|_A = \beta$.

Now by hypothesis there exists a $\theta : A \oplus B \rightarrow A \oplus B$ such that $\nu\theta = \alpha\nu$. Clearly $\theta(A) \subseteq B$. Also if $a \neq 0$ and $\theta(a) = 0$, then $\nu\theta(a) = 0$ whence $\alpha\nu(a) = \alpha(a) = 0$ which contradicts that $\alpha|_A$ is an isomorphism. Thus θ is monic on A . Then we have $A \xrightarrow{\theta} B \xrightarrow{\alpha^{-1}\nu} A$ so that $\alpha^{-1}\nu\theta = \alpha^{-1}\alpha\nu = \text{id}_A$. Hence A is isomorphic to a direct summand of B .

Lemma 2.2. Let $G = D \oplus R$, where D is divisible and R is reduced, be s.p.p. If $D \neq 0$, then R must be bounded.

PROOF. Consider $G = Z(p^\infty) \oplus R \oplus D_1$. Then clearly $G_1 = R \oplus Z(p^\infty)$ is s.p.p. Now if R is unbounded R contains a pure subgroup S such that $R/S \simeq Z(p^\infty)$. Hence by Lemma 2.1 $Z(p^\infty) \simeq$ a direct summand of R which is reduced, a contradiction.

Proposition 2.3. Let C be s.p.p. and let R be pure in C . Then C/R is s.p.p. if and only if the following holds:

(*) For every pure subgroup $L \supseteq R$ and for every endomorphism θ of C such that $\theta(L) \subseteq L$, there exists an endomorphism σ of C such that $\sigma(R) \subseteq R$ and $(\theta - \sigma)(C) \subseteq L$.

PROOF. Assume C/R is s.p.p., L a pure subgroup $\supseteq R$. Let $\eta : C \rightarrow C/R$; $\bar{\eta} : C/R \rightarrow C/L$. Let θ be an endomorphism of C such that

$\theta(L) \subseteq L$. Let $\bar{\theta}$ be the endomorphism on C/L induced by θ . Now by hypothesis there exists $\bar{\sigma} : C/R \rightarrow C/R$ such that $\bar{\eta} \bar{\sigma} = \bar{\theta} \bar{\eta}$.

Since C is s.p.p. there exists a $\sigma : C \rightarrow C$ such that $\eta\sigma = \bar{\sigma}\eta$. Also $\eta\sigma(R) = \bar{\sigma}\eta(R) = 0$. Hence $\sigma(R) \subseteq R$. Further consider $\bar{\eta}\eta(\theta - \sigma)(c) = \bar{\eta}(\eta\theta(c)) - \bar{\eta}(\eta\sigma(c)) = \bar{\eta}(\theta(c) + R) - \bar{\eta}(\bar{\sigma}\eta(c)) = [\theta(c) + L] - \bar{\theta}\bar{\eta}(c + R) = [\theta(c) + L] - \bar{\theta}(c + L) = [\theta(c) + L] - [\theta(c) + L] = 0$ of C/L . Hence $(\theta - \sigma)(c) \in L$ for every c i.e. $(\theta - \sigma)(C) \subseteq L$.

Conversely, let C be an s.p.p. satisfying $(*)$. Then we claim C/R is s.p.p. where R is pure in C . Identifying a given pure quotient of C/R as C/L where L is pure in C containing R , let $\bar{\theta}$ be an endomorphism of C/L , and let $\eta : C \rightarrow C/R$ and $\bar{\eta} : C/R \rightarrow C/L$ be the natural maps. Now since C is s.p.p. there exists a $\theta : C \rightarrow C$ such that $\bar{\theta}\bar{\eta}\eta = \bar{\eta}\eta\theta$. Also it is clear that $\theta(L) \subseteq L$ since $(\bar{\eta}\eta)\theta(L) = \bar{\theta}\bar{\eta}\eta(L) = 0$ of C/L . Hence by $(*)$ there exists a $\sigma : C \rightarrow C$ such that $\sigma(R) \subseteq (R)$ and $(\theta - \sigma)(C) \subseteq L$.

Let $\bar{\sigma}$ be the induced map on C/R . Now consider $\bar{\eta} \bar{\sigma}(c + R) = \bar{\eta}(\sigma(c) + R) = \sigma(c) + L$, and $\bar{\theta}\bar{\eta}(c + R) = \bar{\theta}(c + L) = \theta(c) + L$. Also $(\theta - \sigma)(c) \in L$ for every $c \in C$. Hence $\bar{\eta} \bar{\sigma} = \bar{\theta} \bar{\eta}$. Thus C/R is s.p.p.

Lemma 2.4. *Let G be a reduced s.p.p. group. If B is a basic subgroup of G then $|B| = |G|$.*

PROOF. [Reid] Let B be basic in G and assume $|B| < |G|$. Put $D = G/B$ so that $|D| = |G|$. Since G is s.p.p. and B is pure in G we have for every endomorphism θ of D , and endomorphism α of G such that $\nu\alpha = \theta\nu$, ν the natural map of G onto D . Let $E(G, B) = \{\sigma : \sigma \text{ is an endomorphism of } G \text{ such that } \sigma(B) \subseteq B\}$. Then we have an isomorphism $E(D) \simeq \frac{E(G, B)}{\text{Hom}(G, B)}$. Since G is reduced, restriction gives a monomorphism of $E(G, B)$ into $E(B)$ so that $|E(G, B)| \leq |E(B)|$. But $|E(D)| = 2^{|D|}$ and $|E(B)| = 2^{|B|}$. Hence $2^{|D|} = |E(D)| \leq |E(G, B)| \leq |E(B)| \leq 2^{|B|}$. But by assumption $2^{|B|} < 2^{|D|}$, a contradiction to G.C.H.. This proves the lemma.

Lemma 2.5. *If G is s.p.p. and B is basic in G , then we can assume $r(G) = \text{finr}(G) = r(B) = \text{finr}(B)$.*

PROOF. First we write $G = G' \oplus G''$ where G' is bounded and $r(G'') = \text{finr}(G'')$. [4 §.35]. Then G'' is s.p.p. and G is a direct sum of cyclic groups if G'' is. So we may assume $r(G) = \text{finr}(G)$ to begin with. Similarly for B basic in G , we write $B = B' \oplus B''$ with B' bounded and $r(B'') = \text{finr}(B'')$. Then $G = B' \oplus H$ for some H and we have $r(G) \geq r(H) \geq \text{finr}(H) = \text{finr}(G) = r(G)$. Hence $r(H) = \text{finr}(H)$. Now consider a basic subgroup B_1 of H . Then $B' \oplus B_1$ is basic in G and so $B' \oplus B_1 \simeq B' \oplus B''$. Hence $B'' \simeq$ a basic subgroup of H . Thus we have replaced G by an s.p.p. group H satisfying $r(H) = \text{finr}(H)$ and $r(B) = \text{finr}(B)$ for B basic in H . Further

by Lemma 2.4, $|B| = |H|$ and $r(B) = |B| = r(H)$. Hence the Lemma is proved.

Lemma 2.6. *Let C be a direct sum of cyclic groups with L basic in C . Let B_1 and B_2 be basic in L , such that $C/B_1 \simeq L/B_2$. Then there exists an isomorphism $\phi : C \simeq L$ such that $\phi(B_1) = B_2$.*

PROOF. Since C is a direct sum of cyclic groups and L is basic in C , there is an isomorphism $\psi : C \simeq L$. Put $B'_1 = \psi(B_1)$. Then $L/B_2 \simeq C/B_1 \simeq L/B'_1$. So by Hill's theorem, there is an automorphism λ of L such that $\lambda(B'_1) = B_2$. Consider $\phi = \lambda\psi$. Then ϕ does the work.

Theorem 2.7. *A reduced p -group G is s.p.p. if and only if it is a direct sum of cyclic groups.*

PROOF. Let G be a reduced a s.p.p. p -group. By Lemma 2.5 we may assume $\text{finr}(G) = \text{finr}(B)$ for every basic subgroup B of G . Taking B to be a lower basic subgroup of G and B' to be a lower basic subgroup of B , since $r(G/B) = \text{finr}(G)$ and $r(B/B') = \text{finr}(B)$ and G/B and B/B' are divisible, we have $G/B \simeq B/B'$. Now there exists a pure exact sequence $0 \rightarrow R \rightarrow C \xrightarrow{\eta} G \rightarrow 0$ where C is a direct sum of cyclic groups. Since $|G| = \text{finr}(G)$, G has a pair B_1, B_2 of disjoint lower basic subgroups.

Now we can choose B'_1, B'_2 basic in B_1, B_2 such that $G/B_1 \simeq B_1/B'_1$ and $G/B_2 \simeq B_2/B'_2$. Let $L_1 = \eta^{-1}(B_1)$, $L'_1 = \eta^{-1}(B'_1)$ and $L_2 = \eta^{-1}(B_2)$, $L'_2 = \eta^{-1}(B'_2)$. Clearly L_1, L_2 are basic in C and L'_1, L'_2 are basic in L_1, L_2 and $\frac{C}{L_i} \simeq \frac{L_i}{L'_i}$, $i = 1, 2$. Then by Lemma 2.6, there are isomorphisms $\phi_i : C \simeq L_i$ such that $\phi_i(L_i) = L'_i$. Since $R \subset L'_i \subset L_i$, $i = 1, 2$, by Proposition 2.3 there exist endomorphisms σ_i , $i = 1, 2$ of C such that $\sigma_i(R) \subseteq R$ and $\delta_i(C) = (\phi_i - \sigma_i)(C) \subseteq L_i$.

Let θ_i be the endomorphism of G induced by σ_i . Let K_i be the kernel of θ_i . Then K_i is the image under η of $\{x \in C \text{ such that } \sigma_i(x) \in R\}$. Now if $\sigma_i(x) \in R$, then $(\phi_i - \sigma_i)(x) \in L'_i$. Hence $\phi_i(x) \in L'_i$ and ϕ_i being an automorphism $x \in L_i$. Hence $\eta(x) \in B_i$. Hence $K_i \subseteq B_i$.

Now consider the two endomorphisms θ_1, θ_2 of G into the disjoint basic subgroups B_1 and B_2 with kernel $\theta_i \subseteq B_i$. Define $\theta_1 \times \theta_2 : G \rightarrow B_1 \oplus B_2$ by $(\theta_1 \times \theta_2)(x) = \theta_1(x) + \theta_2(x)$. Now the kernel of $\theta_1 \times \theta_2 = \{x | \theta_1(x) + \theta_2(x) = 0\}$. Also $\theta_1(x) = -\theta_2(x) = 0$ since the θ_i 's map G into B_i . And as the kernel $\theta_i = K_i \subseteq B_i$ we have $x \in B_i, i = 1, 2$. So $x = 0$. Thus G is isomorphic to a subgroup of the direct sum of cyclic groups $B_1 \oplus B_2$ and so is itself a direct sum of cyclic groups.

The converse is well known.

Theorem. **(under G.C.H.) A torsion abelian group $G = \sum G_p$ is*

*The author is grateful to the referee for suggesting the correct form of this theorem and significant changes.

skew pure projective if and only if each G_p is a direct sum of cyclic groups, or a direct sum of a divisible group and a bounded p -group.

PROOF. This easily follows from Remark 1.3, Lemma 2.2 and Theorem 2.7.

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