

On speed of mean convergence of Lagrange and Hermite interpolation based on the roots of Laguerre polynomials

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Dedicated to Professor Béla Gyires on his 80th birthday

The mean convergence of interpolating processes was investigated by many authors after the pioneer work of ERDŐS, TURÁN and BALÁZS [2,4]. See for references on this topic, e.g. [10,12]. We mention first the classical result of ERDŐS and TURÁN [1,2] stating that for a general class of weights, the Lagrange interpolation converges in L^2 -norm on finite interval. For the case of infinite interval, the analogous theorem was proved by BALÁZS and TURÁN [4,5]. In [13] the authors gave an estimate for the speed of L^2 -norm convergence of the Lagrange interpolation with Laguerre abscissas; it was based on a Jackson type theorem corresponding to the Laguerre weight. For the Hermite interpolation on finite interval a general result is given in [1, p. 419].

In [15] JOÓ and SZABADOS proved an L^1 -norm convergence for two interpolating processes based on roots of Laguerre polynomials $L_n^{(\alpha)}(x)$ ($-1 < \alpha \leq 0$). Using the ideas of the present paper, one can extend this result for any $\alpha > -1$.

Here we concern ourselves only to the case of Laguerre nodes, because this contains the ideas for Jacobi and Hermite nodes too. We use the ideas of [2-7].

Denote by $L_n^{(\alpha)}(x) := x^{-\alpha} e^x \frac{1}{n!} [e^x x^{n+\alpha}]^{(n)}$ the Laguerre polynomials for $\alpha > -1$ and by $0 < x_1 < \dots < x_n$ the zeros of $L_n^{(\alpha)}(x)$ (in fact $x_k = x(k, n, \alpha)$; we simplify the notation). Define further

$$\ell_k(x) := \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)'}(x_k)(x - x_k)} \quad (k = 1, \dots, n)$$

the fundamental polynomials of the Lagrange interpolation.

Let $L_n(f, x) := \sum_{k=1}^n f(x_k) \ell_k(x)$ be the Lagrange interpolation of f . For any $f \in C^1(0, \infty)$ the Hermite interpolating polynomial of f is defined as follows:

$$H_n(f, x) := \sum_{k=1}^n \left[f(x_k) \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} + f'(x_k)(x - x_k) \right] \ell_k^2(x).$$

As in [15], introduce the following space:

$$C(\lambda) := \left\{ f \in C[0, \infty) : \lim_{x \rightarrow \infty} f(x) e^{-\lambda x} = 0 \right\},$$

further let

$$E_n(f) := E_n^\lambda(f) := \inf_{P_n \in \Pi_n} \max_{x \geq 0} |f(x) - p_n(x)| e^{-\lambda x}.$$

We shall prove the following

Theorem. Let $\alpha > -1$, $0 < \lambda < 1/2$, then we have the estimates

$$(1) \quad \left(\int_0^\infty x^\alpha e^{-x} |f(x) - L_n(f, x)|^2 dx \right)^{1/2} \leq c \cdot E_{n-1}^\lambda(f) \quad (f \in C[0, \infty))$$

$$(2) \quad \int_0^\infty x^\alpha e^{-x} |f(x) - H_n(f, x)| dx \leq c \cdot E_{2n-1}^\lambda(f') \quad (f \in C^1[0, \infty))$$

($n = 1, 2, \dots$) where c is an absolute constant (independent of f and of n). (Π_n denotes the set of algebraic polynomials of degree at most n).

Lemma. Let $\alpha > -1$ and $0 < \mu < 1/2$, then for any $f \in C(0, \infty)$:

$$(3) \quad \left(\int_0^\infty x^\alpha e^{-x} \cdot L_n(f, x)^2 dx \right)^{1/2} \leq c \cdot \left(\max_{1 \leq k \leq n} |f(x_k)|^2 e^{-\mu x_k} \right)^{1/2}$$

and for any $f \in C^1(0, \infty)$

$$(4) \quad \int_0^\infty x^\alpha e^{-x} |H_n(f, x)| dx \leq c \cdot \max_{1 \leq k \leq n} |f(x_k)| e^{-\mu x_k} + \\ + \bar{c} \cdot \max_{1 \leq k \leq n} |f'(x_k)| e^{-\mu x_k}.$$

PROOF of the Lemma.

Consider a function $g \in C^{2n}(0, \infty)$ with property $g^{(2n)} > 0$ on $(0, \infty)$. Then either $g \geq H_n(g)$ on $(0, \infty)$ or $g \leq H_n(g)$ on $(0, \infty)$. Indeed, if $g(y_1) > H_n(g, y_1)$ and $g(y_2) < H_n(g, y_2)$ then the function $g - H_n(g)$ has at least $2n + 1$ zeros (counted with multiplicity) and then $(g - H_n(g))^{(2n)} = g^{(2n)}$ has also a zero - a contradiction. Choose specially $g(x) = e^{\mu x}$, we get

$$\sum_{k=1}^n \left\{ e^{\mu x_k} \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} + \mu e^{\mu x_k} (x - x_k) \right\} \ell_k^2(x) \leq \\ \leq e^{\mu x} \quad (x \geq 0).$$

Multiplying by $x^\alpha e^{-x}$ and integrating from 0 to ∞ we obtain:

$$(5) \quad \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} \ell_k^2(x) dx \leq \\ \leq \int_0^\infty x^\alpha e^{-(1-\mu)x} dx =: c(\alpha, \mu).$$

Now

$$\int_0^\infty x^\alpha e^{-x} L_n(f, x)^2 dx = \int_0^\infty x^\alpha e^{-x} \left[\sum_{k=1}^n f(x_k) \ell_k(x) \right]^2 dx = \\ = \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n f(x_k)^2 \ell_k^2(x) dx \leq \\ \leq \max_{1 \leq k \leq n} f(x_k)^2 e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \ell_k^2(x) dx = \\ = \max_{1 \leq k \leq n} f(x_k)^2 e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} dx$$

since

$$\begin{aligned} \sum_{k=1}^n e^{\mu x_k} \ell_k^2(x) - \sum_{k=1}^n e^{\mu x_k} \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} \ell_k^2(x) = \\ = \sum_{k=1}^n e^{\mu x_k} \frac{(\alpha + 1 - x_k)(x_k - x)}{x_k} \ell_k^2(x); \end{aligned}$$

further

$$\begin{aligned} \int_0^\infty x^\alpha e^{-x} |H_n(f, x)| dx \leq \int_0^\infty x^\alpha e^{-x} \left| \sum_{k=1}^n f(x_k) \frac{x_k(x_k - \alpha) + x(\alpha + 1 - x_k)}{x_k} \right. \\ \left. \ell_k^2(x) \right| dx + \int_0^\infty x^\alpha e^{-x} \left| \sum_{k=1}^n f'(x_k)(x - x_k) \ell_k^2(x) \right| dx = I_1 + I_2, \end{aligned}$$

by orthogonality

$$\begin{aligned} I_1 \leq \max_{1 \leq k \leq n} |f(x_k)| e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \left[(x_k + |\alpha|) + \left(1 + \alpha \frac{x}{x_k} + x \right) \right] \\ \ell_k^2(x) dx \leq \max_{1 \leq k \leq n} |f(x_k)| e^{-\mu x_k} \left\{ \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{(\mu+1)x_k/2} \ell_k^2(x) dx + \right. \\ \left. + |\alpha| \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \ell_k^2(x) dx + (1 + \alpha) \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \ell_k^2(x) dx + \right. \\ \left. + \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} x_k \ell_k^2(x) dx \right\} \leq \\ \leq \max_{1 \leq k \leq n} |f(x_k)| e^{-\mu x_k} \left\{ 2 \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\frac{\mu+1}{2} x_k} \ell_k^2(x) dx + \right. \\ \left. + (1 + \alpha + |\alpha|) \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} \ell_k^2(x) dx \right\} \leq \\ \leq \max_{1 \leq k \leq n} |f(x_k)| e^{-\mu x_k} \left\{ 2c\left(\alpha, \frac{\mu+1}{2}\right) + (1 + \alpha + |\alpha|) c(\alpha, \mu) \right\} = \\ = \max_{1 \leq k \leq n} |f(x_k)| e^{-\mu x_k} c^*(\alpha, \mu) \end{aligned}$$

and

$$\begin{aligned}
 I_2 &\leq \max_{1 \leq k \leq n} |f'(x_k)| e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} (x + x_k) \ell_k^2(x) dx = \\
 &= 2 \max_{1 \leq k \leq n} |f'(x_k)| e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\mu x_k} x_k \ell_k^2(x) dx \leq \\
 &\leq 2 \max_{1 \leq k \leq n} |f'(x_k)| e^{-\mu x_k} \int_0^\infty x^\alpha e^{-x} \sum_{k=1}^n e^{\frac{\mu+1}{2} x_k} \ell_k^2(x) dx \leq \\
 &\leq 2c \left(\alpha, \frac{\mu+1}{2} \right) \max_{1 \leq k \leq n} |f'(x_k)| e^{-\mu x_k}.
 \end{aligned}$$

Summarizing our estimates we get (4) with $\bar{c} = \max\{c^*(\alpha, \mu), 2c(\alpha, \frac{\mu+1}{2})\}$. $\square$

PROOF of the Theorem.

Let $p_n \in \Pi_{n-1}$ be such that $\max_{x \geq 0} |f(x) - p_n(x)| e^{-\lambda x} \leq E_{n-1}^\lambda(f)$, then applying (3) we get for $\mu = 2\lambda$

$$\begin{aligned}
 &\int_0^\infty x^\alpha e^{-x} |f(x) - L_n(f, x)|^2 dx \leq 2 \int_0^\infty x^\alpha e^{-x} |f(x) - p_n(x)|^2 dx + \\
 &+ 2 \int_0^\infty x^\alpha e^{-x} |L_n(f - p_n, x)|^2 dx \leq 2E_{n-1}^2(f) \int_0^\infty x^\alpha e^{-(1-2\lambda)x} dx + \\
 &+ 2c^2(\alpha, \frac{\mu}{2}) \max_{1 \leq k \leq n} |(f - p_n)(x_k)|^2 e^{-\mu x_k} \leq c_1 E_{n-1}^2(f) + c_2 E_{n-1}^2(f),
 \end{aligned}$$

hence (1) follows. For the proof of (2) pick a polynomial $p_n \in \Pi_{2n-1}$ such that $\max_{x \geq 0} |f'(x) - p'_n(x)| e^{-\lambda x} \leq E_{2n-2}^\lambda(f')$. We can further assume that $f(1) = p_n(1)$, hence

$$\begin{aligned}
 |f(x) - p_n(x)| &= \left| \int_1^x (f'(t) - p'_n(t)) dt \right| \leq \int_1^x |f' - p'_n| \leq E_{2n-2}(f') \cdot \\
 &\cdot \int_1^x e^{\lambda t} dt = O(e^{\lambda x} E_{2n-2}(f')) \quad (x \geq 0)
 \end{aligned}$$

$$\text{i.e. } \max_{x \geq 0} |f(x) - p_n(x)| e^{-\lambda x} \leq c_3 E_{2n-2}(f') \quad (n \geq 1).$$

Hence, applying also (4) for $\mu = \lambda$:

$$\begin{aligned} \int_0^\infty x^\alpha e^{-x} |f(x) - H_n(f, x)| dx &\leq \int_0^\infty x^\alpha e^{-x} |f(x) - p_n(x)| dx + \\ &+ \int_0^\infty x^\alpha e^{-x} |H_n(f - p_n, x)| dx \leq c_3 E_{2n-2}(f') \int_0^\infty x^\alpha e^{-(1-\lambda)x} dx + \\ &+ \bar{c} \max_{1 \leq k \leq n} |f(x_k) - p_n(x_k)| e^{-\lambda x_k} + \bar{c} \max_{1 \leq k \leq n} |f'(x_k) - p'_n(x_k)| e^{-\lambda x_k} = \\ &= O(E_{2n-2}(f')). \quad \square \end{aligned}$$

Remark. For the sake of completeness we have to investigate the quantity $E_n(f)$ defined here. We return to this in a subsequent paper. Taking into account the Stone-Weierstraß Theorem, we have at least: $E_n(f) \rightarrow 0$ ($n \rightarrow \infty$).

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