

On anti-invariant submanifolds in Sasakian manifolds with vanishing contact Bochner curvature tensor

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In a Sasakian manifold, MATSUMOTO and CHŪMAN [4] defined a contact Bochner curvature tensor, which is constructed from the Bochner curvature tensor by the fibering of BOOTHBY and WANG [1] (see also YANO [6]). HASEGAWA and NAKANE [2] and IKAWA and KON [3] have studied a Sasakian manifold with vanishing contact Bochner curvature tensor. YANO [6], [7] studied it in the theory of submanifolds. In this paper we shall give a sufficient condition for an anti-invariant submanifold in a Sasakian manifold with vanishing contact Bochner curvature tensor to be totally geodesic.

1. Preliminaries

Let $\overline{M}^{2r+1}$ be a $(2r+1)$ -dimensional contact metric manifold with the structure $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$, where $\overline{\phi}$ is a linear mapping $T\overline{M} \rightarrow T\overline{M}$ ($T\overline{M}$ is the tangent bundle over $\overline{M}^{2r+1}$), $\overline{\xi}$ is a vector field, $\overline{\eta}$ a 1-form, and $\overline{g}$ a Riemannian metric on $\overline{M}^{2r+1}$, such that

$$\overline{\phi}\overline{\xi} = 0, \quad \overline{\eta}(\overline{\xi}) = 1, \quad \overline{\phi}^2 = -I + \overline{\eta} \otimes \overline{\xi}, \quad \overline{\eta}(\overline{X}) = \overline{g}(\overline{\xi}, \overline{X})$$

$$\overline{g}(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y}), \quad \overline{g}(\overline{X}, \overline{\phi}\overline{Y}) = d\overline{\eta}(\overline{X}, \overline{Y})$$

for any vector fields $\overline{X}$ and $\overline{Y}$ on $\overline{M}^{2r+1}$. If a contact metric manifold $\overline{M}^{2r+1}$ is normal (i.e., $\overline{N} + d\overline{\eta} \otimes \overline{\xi} = 0$, where $\overline{N}$ denotes the Nijenhuis tensor formed with $\overline{\phi}$), $\overline{M}^{2r+1}$ is called a Sasakian manifold. It is well-known that in a Sasakian manifold $\overline{\xi}$ is a Killing vector field.

The contact Bochner curvature tensor $\bar{B}$ of a Sasakian manifold $\bar{M}^{2r+1}$ is given by

$$\begin{aligned}
 \bar{B}(\bar{X}, \bar{Y}) = & \bar{R}(\bar{X}, \bar{Y}) + \frac{1}{m+4}(\bar{Q}\bar{Y} \wedge \bar{X} - \bar{Q}\bar{X} \wedge \bar{Y}) \\
 & + \bar{Q}\bar{\phi}\bar{Y} \wedge \bar{\phi}\bar{X} - \bar{Q}\bar{\phi}\bar{X} \wedge \bar{\phi}\bar{Y} + 2\bar{g}(\bar{Q}\bar{\phi}\bar{X}, \bar{Y})\bar{\phi} \\
 & + 2\bar{g}(\bar{\phi}\bar{X}, \bar{Y})\bar{Q}\bar{\phi} + \bar{\eta}(\bar{Y})\bar{Q}\bar{X} \wedge \bar{\xi} + \bar{\eta}(\bar{X})\bar{\xi} \wedge \bar{Q}\bar{Y} \\
 & - \frac{k+m}{m+4}(\bar{\phi}\bar{Y} \wedge \bar{\phi}\bar{X} + 2\bar{g}(\bar{\phi}\bar{X}, \bar{Y})\bar{\phi}) - \frac{k-4}{m+4}\bar{Y} \wedge \bar{X} \\
 & + \frac{k}{m+4}(\bar{\eta}(\bar{Y})\bar{\xi} \wedge \bar{X} + \bar{\eta}(\bar{X})\bar{Y} \wedge \bar{\xi}),
 \end{aligned}
 \tag{1.1}$$

where $\bar{R}$ is the Riemannian curvature tensor of $\bar{M}^{2r+1}$, $\bar{Q}$ is the Ricci operator of $\bar{M}^{2r+1}$, $k = (\bar{S} + m)/(m + 2)$ ($m = 2r$, and $\bar{S}$ is the scalar curvature tensor of $\bar{M}^{2r+1}$), and $(\bar{X} \wedge \bar{Y})\bar{Z} = \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y}$ (see [4]).

Let M^n be an n -dimensional submanifold of $\bar{M}^{2r+1}$. By N_A ($A = 1, 2, \dots, 2r+1-n$) we denote local mutually orthogonal unit vector fields normal to M^n . Let $\bar{\nabla}$ (resp. ∇) be the Riemannian connection on $\bar{M}^{2r+1}$ (resp. M^n) determined by the metric $\bar{g}$ (resp. the induced metric g). Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_A h_A(X, Y)N_A, \quad \bar{\nabla}_X N_A = -H_A X + \sum_B L_{BA}(X)N_B,$$

where h_A and H_A are the second fundamental forms and L_{BA} the third fundamental forms of M^n . h_A and H_A satisfy $h_A(X, Y) = g(H_A X, Y) = g(X, H_A Y) = h_A(Y, X)$. For any vector field X, Y, Z and W on M^n the Gauss equation is given by

$$\begin{aligned}
 & \bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) \\
 & - \sum_B g(H_B Y, Z)g(H_B X, W) + \sum_B g(H_B X, Z)g(H_B Y, W),
 \end{aligned}
 \tag{1.2}$$

where R is the Riemannian curvature tensor of M^n .

M^n immersed in $\bar{M}^{2r+1}$ is said to be anti-invariant in $\bar{M}^{2r+1}$ if $\bar{\phi}T_x(M^n) \subset T_x(M^n)^\perp$ for each $x \in M^n$, where $T_x(M^n)$ denotes the tangent space of M^n at x and $T_x(M^n)^\perp$ denotes the normal space of M^n at x . Then we see that $\bar{\phi}$ is of rank $2r$, $n \leq r + 1$.

Define $\bar{s}^* = \sum_{i,j=0}^{2r} \bar{g}(\bar{R}(\bar{E}_i, \bar{E}_j)\bar{\phi}\bar{E}_j, \bar{\phi}\bar{E}_i)$ on $\bar{M}^{2r+1}$, where $\{\bar{E}_i\}$ is an orthonormal basis. We need, in the sequel, the following two lemmas

Lemma 1.1. (e.g., [8]) If $\overline{M}^{2r+1}$ is a Sasakian manifold, we have

- (1) $\overline{g}(\overline{Q}\overline{X}, \overline{\xi}) = 2r\overline{\eta}(\overline{X})$
 (2) $\overline{g}(\overline{Q}\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) = \overline{g}(\overline{Q}\overline{X}, \overline{Y}) - 2r\overline{\eta}(\overline{X})\overline{\eta}(\overline{Y})$

Lemma 1.2. ([5]) For any $(2r+1)$ -dimensional Sasakian manifold $\overline{M}^{2r+1}$, we have

$$\overline{s}^* - \overline{S} + 4r^2 = 0.$$

2. Some Lemmas

Lemma 2.1. Let $\overline{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^n (resp. M^{n+1}) is an anti-invariant submanifold of $\overline{M}^{2r+1}$ normal to the structure vector field $\overline{\xi}$ (resp. tangent to the structure vector field $\overline{\xi}$), then we have

$$(r+2)\overline{s}^* = (r+2)\overline{S} - n \sum_{i=1}^n \overline{g}(\overline{Q}\tilde{e}_i, \tilde{e}_i) - 4r^2(r+2),$$

where $\{\tilde{e}_i\}$ is an orthonormal basis of $T_x(M^n)$ (resp. $\{\tilde{e}_i\}$ is an orthonormal basis of $T_x(M^{n+1})$, that is orthogonal to $\overline{\xi}$).

PROOF. Since the contact Bochner curvature tensor of $\overline{M}^{2r+1}$ vanishes, we have

$$\begin{aligned} \overline{g}(\overline{R}(\overline{X}, \overline{Y})\overline{Z}, \overline{W}) &= -\frac{1}{m+4}(\overline{g}(\overline{X}, \overline{Z})\overline{g}(\overline{Q}\overline{Y}, \overline{W}) \\ &\quad - \overline{g}(\overline{Q}\overline{Y}, \overline{Z})\overline{g}(\overline{X}, \overline{W}) - \overline{g}(\overline{Y}, \overline{Z})\overline{g}(\overline{Q}\overline{X}, \overline{W}) \\ &\quad + \overline{g}(\overline{Q}\overline{X}, \overline{Z})\overline{g}(\overline{Y}, \overline{W}) + \overline{g}(\overline{\phi}\overline{X}, \overline{Z})\overline{g}(\overline{Q}\overline{\phi}\overline{Y}, \overline{W}) \\ &\quad - \overline{g}(\overline{Q}\overline{\phi}\overline{Y}, \overline{Z})\overline{g}(\overline{\phi}\overline{X}, \overline{W}) - \overline{g}(\overline{\phi}\overline{Y}, \overline{Z})\overline{g}(\overline{Q}\overline{\phi}\overline{X}, \overline{W}) \\ &\quad + \overline{g}(\overline{Q}\overline{\phi}\overline{X}, \overline{Z})\overline{g}(\overline{\phi}\overline{Y}, \overline{W}) + 2\overline{g}(\overline{Q}\overline{\phi}\overline{X}, \overline{Y})\overline{g}(\overline{\phi}\overline{Z}, \overline{W}) \\ &\quad + 2\overline{g}(\overline{\phi}\overline{X}, \overline{Y})\overline{g}(\overline{Q}\overline{\phi}\overline{Z}, \overline{W}) + \overline{\eta}(\overline{Y})(\overline{g}(\overline{\xi}, \overline{Z})\overline{g}(\overline{Q}\overline{X}, \overline{W}) \\ &\quad - \overline{g}(\overline{Q}\overline{X}, \overline{Z})\overline{g}(\overline{\xi}, \overline{W})) + \overline{\eta}(\overline{X})(\overline{g}(\overline{Q}\overline{Y}, \overline{Z})\overline{g}(\overline{\xi}, \overline{W}) \\ &\quad - \overline{g}(\overline{\xi}, \overline{Z})\overline{g}(\overline{Q}\overline{Y}, \overline{W}))) + \frac{k+m}{m+4}(\overline{g}(\overline{\phi}\overline{X}, \overline{Z})\overline{g}(\overline{\phi}\overline{Y}, \overline{W}) \\ &\quad - \overline{g}(\overline{\phi}\overline{Y}, \overline{Z})\overline{g}(\overline{\phi}\overline{X}, \overline{W}) + 2\overline{g}(\overline{\phi}\overline{X}, \overline{Y})\overline{g}(\overline{\phi}\overline{Z}, \overline{W})) \\ &\quad + \frac{k-4}{m+4}(\overline{g}(\overline{X}, \overline{Z})\overline{g}(\overline{Y}, \overline{W}) - \overline{g}(\overline{Y}, \overline{Z})\overline{g}(\overline{X}, \overline{W}))) \end{aligned} \quad (2.1)$$

$$\begin{aligned}
& - \frac{k}{m+4} (\bar{\eta}(\bar{Y})(\bar{g}(\bar{X}, \bar{Z})\bar{g}(\bar{\xi}, \bar{W}) - \bar{g}(\bar{\xi}, \bar{Z})\bar{g}(\bar{X}, \bar{W})) \\
& + \bar{\eta}(\bar{X})(\bar{g}(\bar{\xi}, \bar{Z})\bar{g}(\bar{Y}, \bar{W}) - \bar{g}(\bar{Y}, \bar{Z})\bar{g}(\bar{\xi}, \bar{W}))).
\end{aligned}$$

If we take a $\bar{\phi}$ -basis $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n, \bar{\xi}, \bar{e}_{n+1}, \bar{e}_{n+2}, \dots, \bar{e}_r, \bar{\phi}\tilde{e}_1, \bar{\phi}\tilde{e}_2, \dots, \bar{\phi}\tilde{e}_n, \bar{\phi}\bar{e}_{n+1}, \dots, \bar{\phi}\bar{e}_r)$, $\bar{s}^*$ is expressed by the following form

$$\begin{aligned}
(2.2) \quad \bar{s}^* = & \sum_{i,j=1}^n \bar{g}(\bar{R}(\tilde{e}_i, \tilde{e}_j)\bar{\phi}\tilde{e}_j, \bar{\phi}\tilde{e}_i) + \sum_{i=1}^n \sum_{j=n+1}^r \bar{g}(\bar{R}(\tilde{e}_i, \bar{e}_j)\bar{\phi}\bar{e}_j, \bar{\phi}\tilde{e}_i) \\
& + \sum_{i=1}^n \sum_{j=1}^n \bar{g}(\bar{R}(\tilde{e}_i, \bar{\phi}\tilde{e}_j)\bar{\phi}^2\tilde{e}_j, \bar{\phi}\tilde{e}_i) \\
& + \sum_{i=1}^n \sum_{j=n+1}^r \bar{g}(\bar{R}(\tilde{e}_i, \bar{\phi}\bar{e}_j)\bar{\phi}^2\bar{e}_j, \bar{\phi}\tilde{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=1}^n \bar{g}(\bar{R}(\bar{e}_i, \tilde{e}_j)\bar{\phi}\tilde{e}_j, \bar{\phi}\bar{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{e}_i, \bar{e}_j)\bar{\phi}\bar{e}_j, \bar{\phi}\bar{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=1}^n \bar{g}(\bar{R}(\bar{e}_i, \bar{\phi}\tilde{e}_j)\bar{\phi}^2\tilde{e}_j, \bar{\phi}\bar{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{e}_i, \bar{\phi}\bar{e}_j)\bar{\phi}^2\bar{e}_j, \bar{\phi}\bar{e}_i) \\
& + \sum_{i=1}^n \sum_{j=1}^n \bar{g}(\bar{R}(\bar{\phi}\tilde{e}_i, \tilde{e}_j)\bar{\phi}\tilde{e}_j, \bar{\phi}^2\tilde{e}_i) \\
& + \sum_{i=1}^n \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{\phi}\tilde{e}_i, \bar{e}_j)\bar{\phi}\bar{e}_j, \bar{\phi}^2\tilde{e}_i) \\
& + \sum_{i=1}^n \sum_{j=1}^n \bar{g}(\bar{R}(\bar{\phi}\tilde{e}_i, \bar{\phi}\tilde{e}_j)\bar{\phi}^2\tilde{e}_j, \bar{\phi}^2\tilde{e}_i) \\
& + \sum_{i=1}^n \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{\phi}\tilde{e}_i, \bar{\phi}\bar{e}_j)\bar{\phi}^2\bar{e}_j, \bar{\phi}^2\tilde{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=1}^n \bar{g}(\bar{R}(\bar{\phi}\bar{e}_i, \tilde{e}_j)\bar{\phi}\tilde{e}_j, \bar{\phi}^2\bar{e}_i)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=n+1}^r \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{\phi}\bar{e}_i, \bar{e}_j)\bar{\phi}\bar{e}_j, \bar{\phi}^2\bar{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=1}^n \bar{g}(\bar{R}(\bar{\phi}\bar{e}_i, \bar{\phi}\bar{e}_j)\bar{\phi}^2\bar{e}_j, \bar{\phi}^2\bar{e}_i) \\
& + \sum_{i=n+1}^r \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{\phi}\bar{e}_i, \bar{\phi}\bar{e}_j)\bar{\phi}^2\bar{e}_j, \bar{\phi}^2\bar{e}_i).
\end{aligned}$$

Thus, using (2.1) in the right hand members of (2.2), we get after some lengthy computation

$$\begin{aligned}
(2.3) \quad \bar{s}^* &= \frac{4r-n+4}{r+2} \sum_{i=1}^n \bar{g}(\bar{Q}\bar{e}_i, \bar{e}_i) + \frac{4r+4}{r+2} \sum_{j=n+1}^r \bar{g}(\bar{Q}\bar{e}_j, \bar{e}_j) \\
& - \frac{k+m}{r+2} \cdot r(2r+1) - \frac{k-4}{r+2} r.
\end{aligned}$$

On the other hand, by Lemma 1.1, we have

$$(2.4) \quad \sum_{j=n+1}^r \bar{g}(\bar{Q}\bar{e}_j, \bar{e}_j) = \frac{1}{2}\bar{S} - \sum_{i=1}^n \bar{g}(\bar{Q}\bar{e}_i, \bar{e}_i) - r.$$

Substituting (2.4) into (2.3), we get our result.

Lemma 2.2. Let $\bar{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^n is an anti-invariant submanifold normal to the structure vector field $\bar{\xi}$ of $\bar{M}^{2r+1}$, then we have

$$\begin{aligned}
4(r+1)(n-1) \sum_{i=1}^n \bar{g}(\bar{Q}\bar{e}_i, \bar{e}_i) &= 4(r+1)(r+2)S - 4(r+1)(r+2) \sum_B (Tr H_B)^2 \\
&+ 4(r+1)(r+2) \sum_B Tr H_B^2 - n(n-1)(6r+8) + n(n-1)\bar{S},
\end{aligned}$$

where S is the scalar curvature tensor of M^n .

PROOF. Taking a $\bar{\phi}$ -basis, we find

$$\begin{aligned}
(2.5) \quad \sum_{i=1}^n \bar{g}(\bar{Q}\bar{e}_i, \bar{e}_i) &= \sum_{i,j=1}^n \bar{g}(\bar{R}(\bar{e}_j, \bar{e}_i)\bar{e}_i, \bar{e}_j) + \sum_{i=1}^n \bar{g}(\bar{R}(\bar{\xi}, \bar{e}_i)\bar{e}_i, \bar{\xi}) \\
&+ \sum_{i=1}^n \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{e}_j, \bar{e}_i)\bar{e}_i, \bar{e}_j) + \sum_{i=1}^n \sum_{j=1}^n \bar{g}(\bar{R}(\bar{\phi}\bar{e}_j, \bar{e}_i)\bar{e}_i, \bar{\phi}\bar{e}_j) \\
&+ \sum_{i=1}^n \sum_{j=n+1}^r \bar{g}(\bar{R}(\bar{\phi}\bar{e}_j, \bar{e}_i)\bar{e}_i, \bar{\phi}\bar{e}_j).
\end{aligned}$$

Here, by (1.2) we have

$$\sum_{i,j=1}^n \bar{g}(\bar{R}(\tilde{e}_j, \tilde{e}_i)\tilde{e}_i, \tilde{e}_j) = S - \sum_B (Tr H_B)^2 + \sum_B Tr H_B^2.$$

Moreover, using (2.1) in any other members on the right hand side of (2.5), we get

$$(2.6) \quad \begin{aligned} & \frac{1}{r+2} \sum_{i=1}^n \bar{g}(\bar{Q}\tilde{e}_i, \tilde{e}_i) + \frac{n}{r+2} \sum_{j=n+1}^r \bar{g}(\bar{Q}\tilde{e}_j, \tilde{e}_j) + S - \sum_B (Tr H_B)^2 \\ & + \sum_B Tr H_B^2 - \frac{n}{2r+4} (4(n-r-1) + k(2r-n+3)) = 0. \end{aligned}$$

Substituting (2.4) into (2.6), we obtain our result.

Lemma 2.3. *Let $\bar{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^{n+1} is an anti-invariant submanifold tangent to the structure vector field $\bar{\xi}$ of $\bar{M}^{2r+1}$, then we have*

$$\begin{aligned} & 4(r+1)(n-1) \sum_{i=1}^n \bar{g}(\bar{Q}\tilde{e}_i, \tilde{e}_i) = 4(r+1)(r+2)S \\ & - 4(r+1)(r+2) \sum_B (Tr H_B)^2 + 4(r+1)(r+2) \sum_B Tr H_B^2 + n(n-1)\bar{S} \\ & - 2n(4r^2 + 3(3+n)r + 4n + 4). \end{aligned}$$

PROOF. Taking a $\bar{\phi}$ -basis, we have (2.5). Here, by (1.2), we get

$$\begin{aligned} \sum_{i,j=1}^n \bar{g}(\bar{R}(\tilde{e}_j, \tilde{e}_i)\tilde{e}_i, \tilde{e}_j) &= S - \sum_B (Tr H_B)^2 + \sum_B Tr H_B^2 \\ &\quad - 2 \sum_{i=1}^n \bar{g}(\bar{R}(\bar{\xi}, \tilde{e}_i)\tilde{e}_i, \bar{\xi}). \end{aligned}$$

From (2.1) and Lemma 1.1, we find

$$\sum_{i=1}^n \bar{g}(\bar{R}(\bar{\xi}, \tilde{e}_i)\tilde{e}_i, \bar{\xi}) = \frac{2n(r+2)}{m+4}.$$

Thus, we have

$$\sum_{i,j=1}^n \bar{g}(\bar{R}(\tilde{e}_j, \tilde{e}_i)\tilde{e}_i, \tilde{e}_j) = S - \sum_B (Tr H_B)^2 + \sum_B Tr H_B^2 - \frac{4n(r+2)}{m+4}.$$

Moreover, using (2.1) in any other members on the right hand side of (2.5), we find

$$(2.7) \quad \begin{aligned} & \frac{1}{r+2} \sum_{i=1}^n \bar{g}(\bar{Q}\tilde{e}_i, \tilde{e}_i) + \frac{n}{r+2} \sum_{j=n+1}^r \bar{g}(\bar{Q}\bar{e}_j, \bar{e}_j) + S - \sum_B (Tr H_B)^2 \\ & + \sum_B Tr H_B^2 - \frac{n}{2r+4} (4(n+1) + k(2r-n+3)) = 0. \end{aligned}$$

Substituting (2.4) into (2.7), we obtain our result.

3. The results

Proposition 3.1. *Let M^{2r+1} be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^n is an anti-invariant submanifold normal to the structure vector field $\bar{\xi}$ of $\bar{M}^{2r+1}$, then we have*

$$(3.1) \quad 4n(r+1)(r+2) \left(S + \sum_B Tr H_B^2 - \sum_B (Tr H_B)^2 \right) = n^2(n-1)(6r+8-\bar{S}).$$

PROOF. By Lemma 1.2, Lemma 2.1 and Lemma 2.2, we have our result.

From (3.1) we get the following.

Proposition 3.2. *Let M^n be a minimal anti-invariant submanifold normal to the structure vector field $\bar{\xi}$ of a Sasakian manifold $\bar{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature S of M^n satisfies $S \geq 0$, then the scalar curvature $\bar{S}$ of $\bar{M}^{2r+1}$ satisfies $\bar{S} \leq 2(3r+4)$.*

Theorem 3.1. *Let M^n be a minimal anti-invariant submanifold normal to the structure vector field $\bar{\xi}$ of a Sasakian manifold $\bar{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature $\bar{S}$ of $\bar{M}^{2r+1}$ satisfies $\bar{S} \geq 2(3r+4)$ and the scalar curvature S of M^n satisfies $S \geq 0$, then M^n is totally geodesic, and $\bar{S} = 2(3r+4)$, $S = 0$.*

PROOF. By our assumption, the left hand side of (3.1) is non-negative and the right hand side non-positive. This completes the proof of the theorem.

Proposition 3.3. Let $\overline{M}^{2r+1}$ be a Sasakian manifold with vanishing contact Bochner curvature tensor. If M^{n+1} is an anti-invariant submanifold tangent to the structure vector field $\bar{\xi}$ of $\overline{M}^{2r+1}$, then we have

$$\begin{aligned}
 & 4(r+1)(r+2)n \sum_B \text{Tr} H_B^2 + 4(r+1)(r+2)nS \\
 (3.2) \quad & = 4(r+1)(r+2)n \sum_B (\text{Tr} H_B)^2 \\
 & + (n-1)(4r(r+1) - n^2)(\bar{S} - (8+6r)) \\
 & + 8(r+1)(r+2)(n^2 + r(2r+1)(n-1)).
 \end{aligned}$$

PROOF. By Lemma 1.2, Lemma 2.1 and Lemma 2.3, we get our result.

Theorem 3.2. Let M^{n+1} be a minimal anti-invariant submanifold tangent to the structure vector field $\bar{\xi}$ of a Sasakian manifold $\overline{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature $\bar{S}$ of $\overline{M}^{2r+1}$ satisfies $\bar{S} \leq 2(3r+4)$ and the scalar curvature S of M^{n+1} satisfies $S \geq \frac{2(n^2+r(2r+1)(n-1))}{n}$, then M^{n+1} is totally geodesic, and $S = \frac{2(n^2+r(2r+1)(n-1))}{n}$, $\bar{S} = 2(3r+4)$.

PROOF. From (3.2), we have the following relation

$$\begin{aligned}
 (3.3) \quad & 4n(r+1)(r+2) \left(\sum_B \text{Tr} H_B^2 + S - \frac{2(n^2 + r(2r+1)(n-1))}{n} \right. \\
 & \left. - \sum_B (\text{Tr} H_B)^2 \right) = (n-1)(4r(r+1) - n^2)(\bar{S} - (8+6r)).
 \end{aligned}$$

This completes the proof of the Theorem.

From (3.3) we have the following

Proposition 3.4. Let M^{n+1} be a minimal anti-invariant submanifold tangent to the structure vector field $\bar{\xi}$ of a Sasakian manifold $\overline{M}^{2r+1}$ with vanishing contact Bochner curvature tensor. If the scalar curvature S of M^{n+1} satisfies $S \geq \frac{2(n^2+r(2r+1)(n-1))}{n}$, then the scalar curvature $\bar{S}$ of $\overline{M}^{2r+1}$ satisfies $\bar{S} \geq 2(3r+4)$.

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