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Linear extensions of partial orders preserving antimonotonicity

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1. Introduction

A classic theorem due to E. SZPILRAJN (see [2]) asserts that any partial order has a linear extension. As a generalization of this result J. SZIGETI and B. NAGY proved in [1]

Theorem A. *Let $(P;p)$ be a partially ordered set and $f : P \to P$ a p–monotone function (i.e. $(a,b) \in p \Longrightarrow (f(a), f(b)) \in p$). Then p can be extended to a linear order r such that f is r–monotone if and only if f is acyclic (i.e. for no $a \in P$ we have $f^n(a) = a$ with $n \in N$, $n > 1$ where $f^1 = f$ and $f^{n+1} = f(f^n)$).*

The aim of this paper is to study some related problems. First we separate the sets where f maps from and to: let $(P;p)$ and $(Q;q)$ be partially ordered sets, $f : P \to Q$ a $p-q$–monotone function (i.e. $(a,b) \in p \Longrightarrow (f(a), f(b)) \in q$). When can p and q be extended to linear orders r and s, respectively, such that f is $r-s$–monotone? The answer is 'always' (Theorem 1) and in fact the proof based on Theorem A is quite easy.

It is more interesting when we are given two functions: $f : P \to Q$ is $p-q$–monotone and $q : Q \to P$ is $q-p$–monotone. By Theorem A, $g \circ f$ and $f \circ g$ must be acyclic if there exist linear extensions r of p and s of q such that f is $r-s$–monotone and g is $s-r$–monotone. It turns out that this trivial necessary condition is also sufficient (Theorem 2). Interestingly, this result contains Theorem A: choose $P = Q$, $p = q$ and $g = \mathrm{id}_p$ and Theorem 1 too: choose g to be any constant function.

In Theorem 3 we change the monotone function of Theorem A for an antimonotone one (i.e. $(a,b) \in p \Longrightarrow (f(b), f(a)) \in p$). We soon notice: if

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p can be extended to a linear order such that f is r–antimonotone then f^2 must be acyclic, moreover, f can have at most 1 fixed point. As another application of Theorem 2, we show that these conditions together are also sufficient.

Finally, in section 4 we propose a generalization of the problems investigated in this paper.

2. Two basic lemmas

We will often use the following two lemmas. For a relation r we denote the smallest transitive relation containing r, i.e. the transitive closure of r by $\bar{r}$.

Lemma B. (cf. J. SZIGETI and B. NAGY [1]). *Let P and Q be two sets with relations $p \subseteq P \times P$, $q \subseteq Q \times Q$ and let $f : P \to Q$ be a function. If for all $(a, b) \in p$ we have $(f(a), f(b)) \in q$ then for all $(a, b) \in \bar{p}$ we have $(f(a), f(b)) \in \bar{q}$.*

The next lemma was proved — in a somewhat different form — by W. T. TROTTER Jr. and J. I. MOORE Jr. in [3].

Lemma C. *Let P be a set with a relation $r \supseteq \{(a, a) \mid a \in P\}$. Then $\bar{r}$ is a partial order if and only if $r \setminus \{(a, a) \mid a \in P\}$ contains no cycle, i.e. a set of the form $\{(a_1, a_2), \ldots, (a_{k-1}, a_k), (a_k, a_1)\}$.*

3. Results

Theorem 1. *Let $(P; p)$ and $(Q; q)$ be partially ordered sets and let $f : P \to Q$ be a p–q–monotone function. Then there are linear extensions r and s of p and q, respectively, such that f is $r-s$–monotone.*

PROOF. Without loss of generality we may suppose $P \cap Q = \emptyset$. Let $T = P \cup Q$, $t = p \cup q$ and define $g : T \to T$ by $g(x) = f(x)$ if $x \in P$, $g(x) = x$ if $x \in Q$.

It is sraightforward to check that $(T; t)$ is a partially ordered set and g is an acyclic t–monotone function. Then, by Theorem A, t can be extended to a linear order t' so that g be t'–monotone. Clearly, $r = P^2 \cap t'$ and $s = Q^2 \cap t'$ will work.

Theorem 2. *Let $(P; p)$ and $(Q; q)$ be partially ordered sets and suppose that $f : P \to Q$ is a $p-q$–monotone function and $g : Q \to P$ is a $q-p$–monotone function. Then there exist linear extensions r of p and s of q such that f is $r-s$–monotone and g is $s-r$–monotone if and only if $g \circ f$ and $f \circ g$ are acyclic.*

Remark. Since $g \circ f$ and $f \circ g$ are acyclic at the same time it would be enough to suppose that $g \circ f$ is acyclic.

PROOF of Theorem 2. Necessity is trivial; we only have to prove sufficiency.

Let E denote the set of pairs (p', q') such that $p \subseteq p' \subseteq P^2$ and $q \subseteq q' \subseteq Q^2$ are partial orders; f is $p' - q'$–monotone and g is $q' - p'$–monotone. With $(p', q') \leq (p'', q'')$ iff $p' \subseteq p''$ and $q' \subseteq q''$, E is a partially ordered set. If

$$C = \{(p_i, q_i) \mid i \in I\} \subseteq E$$

is a chain then

$$\left(\bigcup_{i \in I} p_i, \bigcup_{i \in I} q_i \right) \in E$$

is an upper bound for C. Thus we can apply Zorn's lemma: E has a maximal element, say (r, s), and it is enough to show that r and s are linear orders. Suppose r or s is not linear.

Case 1. For some $a, b \in P$, we have $(a, b) \notin r$, $(b, a) \notin r$ but $(f(a), f(b)) \in s$. Then for

$$r' = r \cup \{(a', b') \in P \mid (a', a), (b, b') \in r\} \supset r$$

we have $(r', s) \in E$, a contradiction.

Case 2. For any $a, b \in P$ with $(a, b) \notin r$ and $(b, a) \notin r$ we have $(f(a), f(b)) \notin s$ and for any $c, d \in Q$ with $(c, d) \notin s$ and $(d, c) \notin s$ we have $(g(c), g(d)) \notin r$.

Let $a, b \in P$ be incomparable elements and define a_i, b_i, c_i, d_i $(i = 1, 2, \dots)$ by

$$\begin{gathered} a_1 = a, \; b_1 = b, \\ c_i = f(a_i), \; d_i = f(b_i), \\ a_{i+1} = g(c_i), \; b_{i+1} = g(d_i). \end{gathered}$$

$$\begin{aligned} \text{Let } \; r_1 &= r \cup \{(a_i, b_i) \mid i = 1, 2, \dots\}, \; r_2 = r \cup \{(b_i, a_i) \mid i = 1, 2, \dots\}, \\ s_1 &= s \cup \{(c_i, d_i) \mid i = 1, 2, \dots\}, \; s_2 = s \cup \{(d_i, c_i) \mid i = 1, 2, \dots\}. \end{aligned}$$

Since $g \circ f$ and $f \circ g$ are acyclic the transitive closure of r_1 or r_2 and the transitive closure of s_1 or s_2 are partial orders. More is true: if $\bar{r}_i$ is a partial order then $\bar{s}_i$ is a partial order too $(i = 1, 2)$. Suppose e.g., that $\bar{r}_1$ is a partial order but $\bar{s}_1$ is not. Then for some sequence of indices $j_1, \dots, j_k$ we must have

$$(d_{j_1}, c_{j_2}), \dots, (d_{j_{k-1}}, c_{j_k}), (d_{j_k}, c_{j_1}) \in s$$

according to Lemma C. Since g is monotone, it follows

$$(b_{i_1}, a_{i_2}), \ldots, (b_{i_{k-1}}, a_{i_k}), (b_{i_k}, a_{i_1}) \in r,$$

where we put $i_t = j_t + 1$ for the sake of simple notation. But then r_1 also contains a cycle, contradicting our assumption that $\bar{r}_1$ is a partial order. By Lemma B, f is $\bar{r}_1$–$\bar{s}_1$–monotone and g is $\bar{s}_1$–$\bar{r}_1$–monotone which means (r, s) is not maximal in E, a contradiction.

Theorem 2 can be considered as a generalization of Theorem A: simply we have to set $P = Q$, $p = q$ and $g = \mathrm{id}_p$ in Theorem 2 to obtain the assertion of Theorem A. Theorem 2 also contains Theorem 1: if we define g to be an arbitrary constant function then $f \circ g$ and $g \circ f$ will be acyclic.

As another application of Theorem 2 we prove

Theorem 3. *Let $(P; p)$ be a partially ordered set and let $f : P \to P$ be a p–antimonotone function. Then there exists a linear extension r of p such that f is p–antimonotone if and only if f^2 is acyclic and f has at most 1 fixed point.*

PROOF. Necessity is trivial; we have to prove sufficiency.

Let E denote the set of partial orders p' on P with the properties $p \le p'$ and f is p'–antimonotone. With the inclusion relation E is a partially ordered set and if $C \subseteq E$ is a chain then $\cup C \in E$ is an upper bound for C. Thus we can apply Zorn's lemma: E has a maximal element, say r. We will show that r is a linear order.

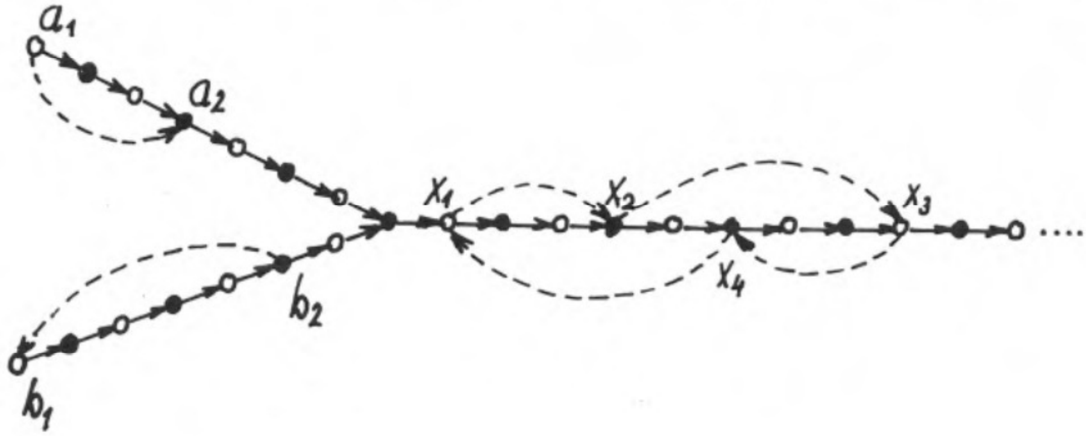
Case 1. f is fixed point free.

First we define an equivalence relation σ on P by

$$a \sigma\, b \iff f^k(a) = f^\ell(b) \text{ for some } k, \ell \in \mathbf{N}^0.$$

Let $\mathcal{C} = \{C \mid C \text{ is a } \sigma\text{–class}\}$. Any $C \in \mathcal{C}$ can be written in the form $C = C_1 \cup C_2$ with $C_1 \cap C_2 = \emptyset$, $f(C_1) = C_2$, $f(C_2) = C_1$.

Observe that if $a_1, b_1 \in C_1$ and $a_2, b_2 \in C_2$ then $(a_1, a_2) \in r$ and $(b_2, b_1) \in r$ simultaneously cannot hold. The diagram describes the case when f is acyclic ($\longrightarrow$ is for f $\dashrightarrow$ is for r, $\circ$ is for C_1 and $\bullet$ is for C_2).



Applying f for the relation $a_1\, r\, a_2$ and for $b_2\, r\, b_1$ again and again we obtain $x_1\, r\, x_2$, $x_3\, r\, x_4$ and $x_2\, r\, x_3$, $x_4\, r\, x_1$, respectively. This means r contains a cycle, a contradiction. If f has a cycle of lenght two, say $f(a) = b$ and $f(b) = a$ for some $a, b \in C$, then in a similar way we get $a\, r\, b$ and $b\, r\, a$ which is a contradiction again.

Next we note that if $C_2 \times C_1 \cap r = \emptyset$ then $C_1 \times C_2 \leq r$. Indeed, now $C_1 \times C_2 \cup r$ cannot contain a cycle. Then by Lemma C, $\overline{C_1 \times C_2 \cup r}$ is a partial order and by Lemma B, f is $\overline{C_1 \times C_2 \cup r}$–antimonotone whence by maximality of r we get $C_1 \times C_2 \subseteq r$. Suppose that for any $C \in \mathcal{C}$ in the above union C_1 denotes the lower part of C, i.e. for which $C_1 \times C_2 \subseteq r$.

Let $C, D \in \mathcal{C}$, $c \in C_1$, $d \in D_2$. If $d\, r\, c$ then $f(c)\, r\, f(d)$ and we have a cycle $d\, r\, c\, r\, f(c)\, r\, f(d)\, r\, d$ in r which is a contradiction. Thus $r \cap D_2 \times C_1 = = r \cap C_2 \times D_1 = \emptyset$ and then it is easy to see that $r \cup C_1 \times D_2 \cup D_1 \times C_2$ cannot contain a cycle. Again by Lemma C and by Lemma B, it follows that $C_1 \times D_2,\ D_1 \times C_2 \subseteq r$.

Finally let

$$
\begin{aligned}
P_1 &= \bigcup_{C \in \mathcal{C}} C_1, \quad & P_2 &= \bigcup_{C \in \mathcal{C}} C_2, \\
p_1 &= r \cap C_1^2, \quad & p_2 &= r^{-1} \cap C_2^2, \\
f_1 &= f \restriction C_1, \quad & f_2 &= f \restriction C_2.
\end{aligned}
$$

Then $f_1 : P_1 \to P_2$ is $p_1 - p_2$–monotone and $f_2 : P_2 \to P_1$ is $p_2 - p_1$–monotone. In wiew of Theorem 2 there are linear extensions r_1 of p_1 and r_2 of p_2 such that f_1 is $r_1 - r_2$–monotone and f_2 is $r_2 - r_1$–monotone. Then $r' = r \cup r_1 \cup r_2^{-1}$ is linear extension of p such that f is r'–antimonotone. In fact, since r was maximal we have $r = r'$.

Case 2. f has a unique fixed point x.

The proof is very similar to that of case 1. We define an equivalence σ on $P \setminus \{x\}$ by

$$a\, \sigma\, b \iff f^k(a) = f^\ell(b) \neq x \text{ for some } k, \ell \in \mathbf{N}^0.$$

Let $\mathcal{C} = \{C \mid C \text{ is a } \sigma\text{–class }\}$ and let

$$
\begin{aligned}
\mathcal{D}' &= \{C \mid C \in \mathcal{C} \text{ and } x \notin f(C)\}, \\
\mathcal{D}'' &= \{C \cup \{x\} \mid C \in \mathcal{C} \text{ and } x \in f(C)\}, \\
\mathcal{D} &= \mathcal{D}' \cup \mathcal{D}''.
\end{aligned}
$$

As in case 1 any $D \in \mathcal{D}$ can be written in the form

$$
\begin{gathered}
D = D_1 \cup D_2, \quad D_1 \cap D_2 = \emptyset, \\
f(D_1) = D_2 \text{ or } D_2 \cup \{x\}, \quad f(D_2) = D_1 \text{ or } D_1 \cup \{x\}.
\end{gathered}
$$

Again, it is easy to see that for any $C, D \in \mathcal{D}$ we have $C_1 \times D_2 \subseteq r$ where C_1 denotes the lower part of C and D_2 denotes the upper part of D.

To be able to use Theorem 2 we let y and z be new elements and define

$$P_1 = \left(\left(\bigcup_{D\in\mathcal{D}} D_1\right) \setminus \{x\}\right) \cup \{y\}, \quad P_2 = \left(\left(\bigcup_{D\in\mathcal{D}} D_2\right) \setminus \{x\}\right) \cup \{z\}$$

and

$$p_1 = (r \cap P_1) \cup P_1 \times \{y\}, \quad p_2 = (r \cap P_2) \cup \{z\} \times P_2)^{-1}.$$

Further, let $f_1 : P_1 \to P_2$ and $f_2 : P_2 \to P_1$ be defined by

$$f_1(u) = \begin{cases} f(u) & \text{if } u \neq y \text{ and } f(u) \neq x, \\ z & \text{otherwise,} \end{cases}$$

$$f_2(u) = \begin{cases} f(u) & \text{if } u \neq z \text{ and } f(u) \neq x, \\ y & \text{otherwise.} \end{cases}$$

If r_1 and r_2 are linear extensions of p_1 and p_2 such that f_1 is r_1-r_2-monotone and f_2 is r_2-r_1–monotone then

$$r' = r_1 \cup (r_1 \setminus P_1 \times \{y\}) \cup (r_2^{-1} \setminus \{z\} \times P)$$

is a linear extension of p and f is r'–antimonotone, completing the proof.

4. A generalization

The above questions can be investigated in the following general context: let (P_i, p_i) $(i \in I)$ be partially ordered sets and let $F_{i,j}$ be a set of monotone (or antimonotone) functions between P_i and P_j $(i, j \in I)$. Find linear extensions of p_i $(i \in I)$ so that all functions remain monotone (or antimonotone).

In its whole generality this problem seems to be rather complicated. On the other hand in the very special case when $\bigcup_{j\in I} F_{i,j}$ consists of at most 1 monotone function for all $i \in I$, we can use the ideas of Theorem 2. Now the necessary and sufficient condition for the existence of linear extensions preserving monotonicity of given functions is: any composition $f_1 \circ \cdots \circ f_k$ with $f_1 \in F_{i_1,j_1}, \ldots, f_k \in F_{i_k,j_k}$ and $i_k = j_1$ be acyclic.

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