Publ. Math. Debrecen
39 / 1–2 (1991), 1–4

Counterexamples to a conjecture about the sum of degrees of irreducible characters

By YAKOV BERKOVICH (Ramat-Gan)

In memory of Professor Lew A. Kaluzhnin

Let $\mathrm{Irr}(G)$ be the set of all irreducible complex characters of a finite group G. If $k(G)$ is the class number of G, then $|\mathrm{Irr}(G)| = k(G)$. Let $T(G) = \sum_{\chi\in \mathrm{Irr}(G)} \chi(1)$, $f(G) = |G|^{-1} \times T(G)$.

If $H \leq G$ then $f(H) \geq f(G)$ (this is an easy corollary of Frobenius reciprocity). Under some suppositions on degrees of irreducible characters the following inequality holds (see [1]): $f(G/H) \geq f(G)$ for all normal subgroups H in G (see [1, Lemma 2.6]). The following conjecture was posed (see question 4.3 in [1]):

$(*)$ If H is normal in G then $f(G/H) \geq f(G)$

In this note we give counterexamples to $(*)$.

In the sequel P is the non-abelian group of order p^3 and exponent $p > 2$. Then

$$P = \langle x, y, c \,|\, x^p = y^p = c^p = [x, c] = [y, c] = 1,\ c = [x, y]\rangle\,.$$

Let A be the set of all $\alpha \in \mathrm{Aut}(P)$ such that $\alpha|_{Z(P)} = \mathrm{id}$. If $\varphi \in \mathrm{Aut}(P)$,

$$\varphi(x) = x^{\alpha_1} y^{\beta_1} c^{\gamma_1},\ \ \varphi(y) = x^{\alpha_2} y^{\beta_2} c^{\gamma_2},$$

then $\varphi \in A$ iff $\alpha_1\beta_2 - \beta_1\alpha_2 = 1$. Hence the special linear group $\mathrm{SL}(2, p)$ is a subgroup of A.

Consider the semi-direct product $H = \mathrm{SL}(2, p)[P$ with the core P such that $Z(H) = Z(P) = \langle c\rangle$ and $\mathrm{SL}(2, p)$ acts faithfully on $P/Z(P)$.

Since $\mathrm{Irr}(P)$ contains exactly $p - 1$ non-linear characters and for $F < \mathrm{SL}(2, p)$ we have $|\mathrm{SL}(2, p) : F| \geq p$ and so

(1) All non-linear characters from $\mathrm{Irr}(P)$ are H–invariant. In particular, if $P \le G \le H$, then all these characters are G–invariant.

All our counterexamples $G = G^p$ satisfy $P < G = G^p < H$, $P \in \mathrm{Syl}_p(G)$. Let $G = G^p = F^p[P = F[P$. By above $F > 1$ and F is a p–complement in G. Since $|\mathrm{SL}(2,p)| = p(p^2-1)$ we have

(2) $\mathrm{Index}|G : P| = |F|$ divides $p^2 - 1$.

By construction $Z(G) = Z(P)$. Let $\bar{G} = G/Z(P)$.

(3) $\bar{G}$ is a Frobenius group with the core $\bar{P}$.

PROOF. Suppose that $\sigma \in F$ and $C_{\bar{P}}(\bar{\sigma}) > 1$. By Maschke's Theorem $\bar{\sigma}$ is represented in an appropriate basis of the linear $GF(p)$–space $\bar{P}$ by the matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. Since $\det \bar{\sigma} = 1$ then $a = 1$, $\bar{\sigma} = 1$, and $\bar{G}$ is a Frobenius group.

(4) $$k(G) = k(\bar{G}) + (p-1)k(F).$$

PROOF. Partition the elements of G into four subsets:

$$G_1 = Z(P)^{\#} = Z(G) - \{1\},$$
$$G_2 = \bigcup_{x \in G} x^{-1}Fx,$$
$$G_3 = P - Z(P),$$
$$G_4 = G - (G_1 \cup G_2 \cup G_3).$$

Then G_1 is the join of $p-1$ one-element G–classes, G_2 is the join of $k(F)$ G–classes. If $x \in P - Z(P)$ then $|C_G(x)| = p^2$ by (3); so $|G : C_G(x)| = p|F|$, so G_3 is the join of

$$\frac{|P - Z(P)|}{p|F|} = \frac{p^2-1}{|F|}$$

G–classes.

Let $x \in G_4$. Then $x = f\pi$, $f \in F$, $\pi \in P$. Since $x \notin G_2$, $0(x)$, the order of x, does not divide $|F|$ by the Schur–Zassenhaus Theorem. Since $x \notin G_1 \cup G_3$ we have $0(x) \ne p$. Thus $f \ne 1 \ne \pi$. Now (3) implies that $0(\bar{x})$ divides $|F|$; here $\bar{x}$ is the image of x under the homomorphism $G \to \bar{G} = G/Z(P)$. Hence $\pi \in Z(P)^{\#} = Z(P) - \{1\}$. Suppose that $f\pi$ is conjugate to $f_1\pi_1$ in G, $f_1 \in F$, $\pi_1 \in Z(P)$. Then there exists $y \in G$ such that $(f\pi)^y = f_1\pi_1$, and $\pi = \pi^y = \pi_1$, $f^y = f_1$. Hence $f_1 \in F \cap F^y$. Since F is a TI–subgroup of G and $f_1 \ne 1$ we have $F = F^y$, $y \in N_G(F) = F \times Z(P)$, and we see that f and f_1 are conjugate in F. Hence G_4 is the union of $(p-1)(k(F)-1)$ G–classes.

Thus $k(G) = p - 1 + k(F) + \frac{p^2-1}{|F|} + (p-1)((k(F)-1) = {}$
$= k(\bar{G}) + (p-1)k(F)$.

It follows from (3) and the properties of irreducible characters of Frobenius groups that

(5) $$T(\bar{G}) = T(F) + p^2 - 1.$$

Compute $T(G)$. Let $\mathcal{M} = \{\chi \in \mathrm{Irr}(G) | Z(P) \not\leq \ker\chi\}$. Since $Z(P)$ is the only minimal normal subgroup of G, $\mathcal{M}$ is the set of all faithful irreducible characters of G, so that

$$T(G) = T(\bar{G}) + \sum_{x \in \mathcal{M}} \chi(1).$$

In view of (5) it remains to compute $\sum^0 = \sum_{\chi \in \mathcal{M}} \chi(1)$.

Suppose that $\chi \in \mathcal{M}$. Since P is non-abelian and χ is faithful, by Clifford's Theorem and (1) one obtains $\chi_P = e_\chi \psi$ where ψ is a non-linear irreducible character of P. In particular, p divides $\psi(1)$. Since $gcd(|P|,|G:P|)= 1$, ψ has an extension $\psi^0 \in \mathrm{Irr}(G)$ [2, Cor. 8.16]. Since $\psi^0_P = \psi$, we have $\psi^G = \psi^0 \varrho_{G/P}$ where $\varrho_{G/P}$ is the regular character of G/P, and irreducible components of ψ^G have a form $\alpha\psi^0$ where $\alpha \in \mathrm{Irr}(G/P)$, and $\alpha \to \alpha\psi^0$ is the injection of $\mathrm{Irr}(G/P)$ on the set of the irreducible components of ψ^G [2, Cor. 6.17]. Suppose that $\psi_1 \neq \psi$ is a non-linear character in $\mathrm{Irr}(P)$, ψ_1^0 is a certain extension of ψ_1 on G, $\alpha_1\psi_1^0$ is an irreducible component of the induced character ψ_1^G and $\alpha_1 \in \mathrm{Irr}(G/P)$. Since $(\alpha\psi^0)_P = \alpha(1)\psi \neq \alpha_1(1)\psi_1 = (\alpha_1\psi_1^0)_P$ then $\alpha_1\psi_1^0 \neq \alpha\psi^0$. So $\langle \psi^G, \psi_1^G \rangle = 0$ and

$$\sum\nolimits^0 = \sum_{\chi \in \mathcal{M}} \chi(1) = \left(\sum_{\substack{\chi \in \mathrm{Irr}(P) \\ \chi(1)=p}} \chi(1) \right) \cdot \left(\sum_{\alpha \in \mathrm{Irr}(G/P)} \alpha(1) \right) = \tag{6}$$
$$= p(p-1)T(F).$$

Thus

$$T(G) = T(\bar{G}) + p(p-1)T(F) = p^2 - 1 + (p^2 - p + 1)T(F). \tag{7}$$

Now

$$f(\bar{G}) = \frac{T(\bar{G})}{|\bar{G}|} = \frac{T(F) + p^2 - 1}{p^2|F|},$$
$$f(G) = \frac{T(G)}{|G|} = \frac{(p^2 - p + 1)T(F) + p^2 - 1}{p^3|F|},$$

$$f(G) - f(\bar{G}) = \frac{(p-1)^2}{p^3|F|}[T(F) - (p+1)]. \tag{8}$$

Hence

(9) $G = F^p[P$ is a counterexample to $(*)$ iff $T(F^p) > p+1$.

Now we give counterexamples to $(*)$.

1°. $F^p = Q(8)$, $G = Q(8)[P < H$.

By (9), $p+1 < T(F^p) = T(Q(8)) = 6$, $p = 3$, $|G| = 2^3 \cdot 3^3$,
Hence in this series our group of order $2^3 \cdot 3^3$ is the only counterexample to $(*)$.

2°. $p > 3$, $F^p = SL(2,3)$.

Then $p+1 < T(F^p) = 12$, $p = 5$ or 7. Thus in this series only two groups G^5 and G^7 are counterexamples to $(*)$.

3°. Let $p \equiv \pm 1 \pmod 8$, $F^p = \hat{S}_4$ where $\hat{S}_4$ is the subgroup of order 48 in $SL(2,p)$; obviously $\hat{S}_4$ is a representation group of S_4, symmetric group of degree 4. Since $p+1 < T(F^p) = T(\hat{S}_4) = 18$ then $p = 7$ and G^7 is the only counterexample in this series.

4°. Let $p \equiv \pm 1 \pmod 5$, $F^p = SL(2,5)$. Since $p+1 < T(F^p) = T(SL(2,5)) = 30$ we have $p = 11$ or 19. Thus in this series only two groups G^{11} and G^{19} are counterexamples to $(*)$.

5°. Suppose that $C(m)$ denotes the cyclic group of order m. Let $p > 3$, and let $F^p = C(4)[C(3)$ be non-abelian. Then $p+1 < T(F^p) = 8$. Hence $p = 5$ and G^5 is the only counterexamples to $(*)$ in this series.

6°. Let F^p be a subgroup of order $2(p+1)$ with a cyclic subgroup of index 2, and $F^p < SL(2,p)$. Then $T(F^p) = p+3 > p+1$. Hence all members G^p of this series are counterexamples to $(*)$.

It is interesting to find such a pair $H \triangleleft G$ for which $f(G/H) < f(G)$ and $H \not\leq Z(G)$. I do not know any counterexample to $(*)$ which is a p-group.

References

[1] K. G. Nekrasov and Ya. G. Berkovich, Finite groups with large sums of degrees of irreducible characters, *Publ. Math. (Debrecen)* **33** (1986), 333–354. (Russian)

[2] I. M. Isaacs, Character Theory of Finite Groups, *Academic Press* (1976), *New York*.

YAKOV BERKOVICH
BAR-ILAN UNIVERSITY
52900 RAMAT-GAN
ISRAEL

(Received July 17, 1987)