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On an arithmetical property of $\sqrt{2}$

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Deditaced to Professor Paul Erdős for his 75th birthday

In [2] it is proved that for every $q \in (1,2)$ excepted a set of first category there exists an expansion $1 = \sum_i q^{-n_i}$ of 1 such that $\limsup_{i\to\infty} (n_{i+1} - n_i) = \infty$. Let $\{y\}$ denote the fractional part of the real number y. We shall prove the following

Theorem.
(i) For almost every $x \in \mathbf{R}$,
(ii) For every $x \in \mathbf{R}$ excepted a set of first category we have

$$\inf_n \left(\{2^n x\} + \left\{ 2^n (1-x)/\sqrt{2} \right\} \right) = 0. \tag{1}$$

Remarks.
1. (1) means that in the dyadic expansions of x and of $\frac{1-x}{\sqrt{2}}$ there are "arbitrarily long gaps at the same places" i.e. if

$$x = \sum_{k=0}^{N} \bar{a}_k 2^k + \sum_{k=1}^{\infty} a_k/2^k, \quad \frac{1-x}{\sqrt{2}} = \sum_{k=0}^{M} \bar{b}_k 2^k + \sum_{k=1}^{M} b_k/2^k, \tag{2}$$

a_n, $\bar{a}_n$, b_n, $\bar{b}_n \in \{0,1\}$ then for every $m \in \mathbf{N}$ there is an n_0 such that

$$a_{n_0+1} = \cdots = a_{n_0+m} = 0 = b_{n_0+1} = \cdots = b_{n_0+m}.$$

2. Taking an $x \in (0,1)$ such that (1) holds we obtain from (2) that

$$1 = \sum 1/\left(\sqrt{2}\right)^{n_i}$$

with $\limsup_{i\to\infty} (n_{i+1} - n_i) = \infty$.

PROOF. Let

$$A_\varepsilon := \left\{ x \in \mathbf{R} \ : \ \inf_n \left(\{2^n x\} + \{2^n(1-x)/\sqrt{2}\,\} \right) < \varepsilon \right\}, \quad \varepsilon > 0 .$$

We prove that A_ε contains some dense open set and $mes\,(\mathbf{R} \setminus A_\varepsilon) = 0$ for any fixed $\varepsilon > 0$. Denote by Z the set of integers. It is well known (see [1], p. 58) that the mod 1 distribution of the set $\{r/\sqrt{2} \ : \ r \in Z\}$ is uniform in the following sense : for every $0 < \varepsilon < 1$ there exists a natural number $X(\varepsilon)$ such that any subsegment of [0,1] of legth at least $\varepsilon/4$ contains mod 1 a number $r/\sqrt{2}$ of any sequence $t/\sqrt{2}, \dots, (t+s)/\sqrt{2}$ if $s > X(\varepsilon)$; $t \in Z$. Consider some $r \in Z$ satisfying

$$\||\, r/\sqrt{2} - \varepsilon/4 \,\|| < \varepsilon/8 , \tag{3}$$

where $\||\, y \,\||$ denotes the distance of the real number y from the nearest integer. Let n be a natural number, then we claim that $I := \left[1 - \frac{1}{2^n}\, , \, 1 - \frac{1}{2^n} + (\varepsilon\sqrt{2}\,/\,8 \cdot 2^n)\right] \subset A_\varepsilon$. Indeed, if $y \in I$ then $\{2^n y\} \leq \varepsilon\sqrt{2}/8 < \varepsilon/4$ and because of $\frac{r}{\sqrt{2}} - \frac{\varepsilon}{8} \leq 2^n(1-y)/\sqrt{2} \leq r/\sqrt{2}$ we get $\{2^n(1-y)/\sqrt{2}\} < \frac{3\varepsilon}{8}$. We see that for any natural n, any segment of legth $\geq \frac{X(\varepsilon)+3}{2^n}$ contains a subsegment of legth $\varepsilon\sqrt{2}/8 \cdot 2^n$ belonging to A_ε. Indeed, we call $r \in Z$ good if (3) holds. We know that the distance between two consecutive good numbers is not greater than $X(\varepsilon) + 2$, hence the distance of the corresponding left endpoints of the intervals I is $\geq \frac{X(\varepsilon)+2}{2^n}$, and so any segment of legth $\frac{X(\varepsilon)+2}{2^n} + \varepsilon\frac{\sqrt{2}}{8}\frac{1}{2^n} < \frac{X(\varepsilon)+3}{2^n}$ contains indeed a segment of legth $\varepsilon\frac{\sqrt{2}}{8}\frac{1}{2^n}$ from A_ε. Hence A_ε contains a dense open set i.e. the complement of $\bigcap_{\varepsilon>0} A_\varepsilon$ is of first category, further the complement of A_ε has no density point consequently mes $(\mathbf{R} \setminus A_\varepsilon) = 0$ i.e. the complement of $\bigcap_{\varepsilon>0} A_\varepsilon$ has measure zero. The theorem is proved. $\square$

The method of the proof works also for $\sqrt[n]{2}$ $(n > 2)$.

Problem. What can we say about $\sqrt{3}$?

References

[1] J. W. S. CASSELS, An introduction to diophantine approximation, *Cambridge University Press*, (1957).

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