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Steinhaus Theorem in Summability for P–adic Fields

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Dedicated to the memory of Professor C. T. Rajagopal

Abstract. A classical theorem of STEINHAUS [11] in summability theory asserts that if C is the complex field and $A = (a_{nk})$ is a regular infinite matrix with coefficients in C, then A cannot sum all sequences of 0's and 1's. The purpose of this note is to give a short proof of the above theorem for p–adic fields Q_p and for local fields.

Introduction

After MONNA [3] gave a complete characterization of regular infinite matrices with coefficients in a complete non–archimedean field K, the investigations in summability theory started developing over complete non–archimedean fields. The works cited in [3], [4], [5], [6], [7], [8], [9], [10] represent the so called non–archimedean summability theory. In his thesis [4] NATARAJAN gave a detailed investigation of Steinhaus' Theorem from different points of view. The purpose of this paper is to give a short and direct proof of the above theorem for p–adic fields Q_p and extend the result to cover local fields.

Throughout let K denote a complete non–archimedean valued field. Let $A = (a_{nk})$ $n = 1, 2, \ldots;$ $k = 1, 2, \ldots$ be an infinite matrix with coefficients in K. Given a sequence $x = \{x_n\}$ in K, the sequence $y = \{y_n\}$ is defined by the procedure

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k \tag{1}$$

The matrix A is said to be regular if whenever $x = \{x_n\}$ converges to some L in K, the transformed sequence $y = \{y_n\}$ also converges to the very same L. In [3] the following conditions are given as necessary and sufficient for A to be regular

1) $\operatorname{Sup}_{n,k} |a_{nk}| < \infty$.
2) $\lim(a_{nk}) = 0$ as $n \to \infty$ for all $k = 1, 2 \ldots$.
3) $\mathrm{Lt}\,(\sum_{k=1}^{\infty} a_{nk}) = 1$ as $n \to \infty$.

The following **Theorem of Osgood** is needed in the sequel. A proof of the above theorem is discussed in STROMBERG [12] in page 120.

Let X be an arbitrary topological space and Y an arbitrary metric space with metric ϱ. Suppose $\{F_n\}$ is a sequence of continuous functions from X to Y, converging to the limit function $F : X \to Y$ pointwise. Then there exists a set $E \subseteq X$ of first category such that F is continuous at each $p \in X - E$. In particular if X is a complete metric space then F is continuous at every point of a dense subset of X.

In the sequel Q_p will denote the p–adic field and Δ denote the collection of all sequences of 0's and 1's from Q_p. For $x = \{x_n\}$ and $y = \{y_n\}$ in Δ we define a metric by setting

$$d(x,y) = \operatorname*{Sup}_n |\, p^n(x_n - y_n)| \tag{2}$$

Then (Δ, d) is a metric space and is complete. (Completion of Δ is really essential for the investigation later on as Δ is not empty). We also note that for $x, y \in \Delta$

$$d(x,y) \leq \sum_{n=1}^{\infty} |\, p^n(x_n - y_n)|\,.$$

Theorem 1. *A regular infinite matrix method $A = (a_{nk})$ over Q_p cannot sum all sequences of 0's and 1's in Q_p.*

PROOF. As A is regular, for each n the functional

$$F_n(x) = \sum_{k=1}^{\infty} a_{nk}\, x_k \tag{3}$$

is well defined for all $x = \{x_n\} \in \Delta$ and further it is a continuous function with values in Q_p, in view of the fact

$$|F_n(x) - F_n(y)| \leq \sum_{k=i+1}^{\infty} |a_{nk}| < \varepsilon \tag{4}$$

if $d(x,y) < |p^i|$ for sufficiently large i. Suppose A sums all sequences in Δ then

$$F(x) = \lim F_n(x) \text{ as } n \to \infty \tag{5}$$

is defined for all $x \in \Delta$. We further note that any open ball around any point of Δ contains two points z and w where z is eventually all 1's and w is eventually all 0's. The regularity of A implies that

$$|F(z) - F(w)| = 1 \tag{6}$$

Hence the above argument shows that F is discontinuous at each point of Δ. However by Theorem of Osgood, F must be continuous atleast at one point of Δ. The above contradiction shows the validity of the claim made in the Theorem.

Remark 1. In the above Theorem 1, the only thing that was used relating to the p-adic field was p^n. It is a p-adic null sequence. By a local field we will mean a totally disconnected, non-trivial complete locally compact field K. Replacing $\{p^n\}$ by any other suitable null sequence the same result is seen to be valid for any local field, as well as for any arbitrary non-archimedean complete, non-trivial valued field K.

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