

$f(3,1)$ -Finsler structures and their integrability

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*Dedicated to Professor Dr. Lajos Tamássy on the occasion
of his 65 th birthday*

In [11] K. YANO has unified the notions of almost complex structure and almost contact structure by considering on an n -dimensional manifold M a tensor field f of type $(1,1)$ such that $f^3 + f = 0$ and the rank r of f is everywhere a constant even numbers. This structure has been called an f -structure (or an $f(3,1)$ -structure). The theory of integrability of this structure [4], [9], is given in [8] using f -connections.

The main purpose of the present paper is to introduce the notion of $f(3,1)$ -Finsler structure on M and to study this structure with the method used by R. MIRON [7].

Terminology and notations belongs to M. MATSUMOTO [5] and R. MIRON [6].

1. Preliminaries

Let M be a C^∞ -differentiable manifold, paracompact with n dimensions, let $T(M) = (TM, p, M)$ be its tangent bundle, and let N be a non-linear connection on $T(M)$. We denote by (x^i, y^i) ($i, j, k, \dots, = 1, 2, \dots, n$) the canonical coordinates on TM . Then $\delta_i = \partial_i - N_i^k \dot{\partial}_k$ ($\partial_i = \frac{\partial}{\partial x^i}$, $\dot{\partial}_k = \frac{\partial}{\partial y^k}$, $\delta_i = \frac{\delta}{\delta x^i}$) is a local basis of horizontal distribution N , and $\dot{\partial}_i$ is a local basis of the vertical distribution $(TM)^v$. The dual basis is $(dx^i, \delta y^i)$ where $\delta y^i = dy^i + N_k^i dx^k$. We have

$$(1.1) \quad [\delta_j, \delta_k] = R_{jk}^i \dot{\partial}_i, \quad [\delta_j, \dot{\partial}_k] = (\dot{\partial}_k N_j^i) \dot{\partial}_i, \quad [\dot{\partial}_j, \dot{\partial}_k] = 0,$$

$$(1.2) \quad \delta_k N_j^i - \delta_j N_k^i = R_{jk}^i, \quad \dot{\partial}_k N_j^i - \dot{\partial}_j N_k^i = t_{jk}^i,$$

where R_{jk}^i and t_{jk}^i are the curvature and torsion fields of N .

A Finsler connection on M is a triad $FT = (N, F, C)$, where N is a non-linear connection on M , and F respectively C are the h - and v -connection coefficients, given by

$$(1.3) \quad \begin{aligned} \nabla_{\delta_k} \delta_j &= F_{jk}^i \delta_i, & \nabla_{\delta_k} \dot{\partial}_j &= F_{jk}^i \dot{\partial}_i, \\ \nabla_{\dot{\partial}_k} \delta_j &= C_{jk}^i \delta_i, & \nabla_{\dot{\partial}_k} \dot{\partial}_j &= C_{jk}^i \dot{\partial}_i. \end{aligned}$$

With $|$ and $|$ we denote the h - and v -covariant derivatives with respect to FT .

The torsion Finsler tensor fields of FT will be denoted by: T_{jk}^i , N_{jk}^i , C_{jk}^i , P_{jk}^i , S_{jk}^i , and the curvature Finsler tensor fields of FT will be denoted by: $R_j^i{}_{kh}$, $P_j^i{}_{kh}$, $S_j^i{}_{kh}$.

2. $f(3, 1)$ -Finsler structures and $f(3, 1)$ -Finsler connections

Let M be an n -dimensional differentiable manifold of class C^∞ , and let $x = (x^i)$ and $y = (y^i)$ denote a point of M and a supporting element respectively.

Definition 2.1. A Finsler tensor field $f(x, y) \neq 0$ of type $(1, 1)$ and of class C^∞ is called an $f(3, 1)$ -Finsler structure of index K , if it satisfies

$$(2.1) \quad f^3 + f = 0, \quad \text{rank } \|f(x, y)\| = n - K = 2p,$$

where K, p are integers and $0 \leq K < n$.

Proposition 2.1. For any $f(3, 1)$ -Finsler structure of index K , the operators $h(x, y)$, $v(x, y)$ given by

$$(2.2) \quad h = -f^2, \quad v = f^2 + I,$$

I denoting the identity operator, are complementary projection operators applied to the tangent bundle of M .

Now we denote by $\mathcal{H}$ and $\mathcal{V}$ the complementary distributions corresponding to the projection operators h and v respectively, and we have: $\dim \mathcal{H}_{(x, y)} = n - K$, $\dim \mathcal{V}_{(x, y)} = K$.

Proposition 2.2. An f satisfying the relation (2.1), acts on $\mathcal{H}$ as an almost complex Finsler operator and on $\mathcal{V}$ as a null operator.

In fact, we have

$$(2.3) \quad \begin{cases} fh = hf = f, & fv = vf = 0 \\ f^2h = -h, & f^2v = 0. \end{cases}$$

Remark 2.1. If the rank of $f(x, y)$ is n , then $h = I$ and $v = 0$ and $f(x, y)$ satisfies: $f^2 = -I$. Consequently the $f(3, 1)$ -Finsler structure of minimum index (null index) is an almost complex Finsler structure (cf. with G. B. RIZZA [10], I. ICHIJYO [3], R. MIRON [7], etc.). In this case the dimension n must be even.

Remark 2.2. If the $f(3,1)$ -Finsler structure is of *index 1*, then $\mathcal{H}$ is $(n-1)$ -dimensional, and $\mathcal{V}$ is one-dimensional. Consequently if we denote the local components of $f(x, y)$, $h(x, y)$ and $v(x, y)$ by $f_j^i(x, y)$, $h_j^i(x, y)$ and $v_j^i(x, y)$ ($i, j, \dots = 1, 2, \dots, n$) respectively, then $v_j^i(x, y)$ must have the form: $v_j^i = \eta_j \xi^i$, where $\eta(x, y)$ and $\xi(x, y)$ are covariant and contravariant Finsler vector fields respectively. From the relations (2.2) and (2.3) we have

$$f_r^i f_j^r = -\delta_j^i + \eta_j \xi^i, \quad f_j^i \xi^j = 0, \quad \eta_i f_j^i = 0, \quad \eta_i \xi^i = 1.$$

Thus an $f(3,1)$ -Finsler structure of index 1 is equivalent to an almost contact Finsler structure [2].

Definition 2.2. We shall call the Finsler tensor fields Q_{ij}^{rs} and Q_{ij}^{rs} given by

$$(2.4) \quad \begin{cases} Q_{ij}^{rs} = \frac{1}{2}(\delta_i^r \delta_j^s - \delta_i^r v_j^s - v_i^r \delta_j^s - f_i^r f_j^s + 3v_i^r v_j^s), \\ Q_{ij}^{rs} = \frac{1}{2}(\delta_i^r \delta_j^s + \delta_i^r v_j^s + v_i^r \delta_j^s + f_i^r f_j^s - 3v_i^r v_j^s), \end{cases}$$

the *Obata operators* of the $f(3,1)$ -Finsler structure.

These operators have the symmetry $Q_{ij}^{rs} = Q_{ij}^{sr}$ ($\alpha = 1, 2$), and act on a Finsler tensor field K of type $(1,2)$ as $(QK)_{jk}^i = Q_{js}^{ri} K_{rk}^s$ ($\alpha = 1, 2$). Since $(QQK)_{jk}^i = Q_{js}^{ri} Q_{rp}^{ts} K_{tk}^p$, the product QQ is defined by $(QQ)_{jp}^{ti} = Q_{js}^{ri} Q_{rp}^{ts}$ ($\alpha, \beta = 1, 2$).

Proposition 2.3. Q_1, Q_2 are the supplementary projectors on the module $\tau_{\frac{1}{2}}$ of the tensor fields of type $(1,2)$:

$$(2.5) \quad Q_1 + Q_2 = I, \quad Q_\alpha^2 = Q_\alpha, \quad Q_1 \cdot Q_2 = Q_2 \cdot Q_1 = 0 \quad (\alpha = 1, 2).$$

The proof is elementary.

Proposition 2.4. $Q_2 X = 0$ has solutions, and its general solution is given by $X = Q_1 Y$, where $Y \in \tau_1^2$ is arbitrary.

An important problem concerning an $f(3,1)$ -Finsler structure on M is to determine the existence of arbitrary Finsler connections with respect to which $f_j^i(x, y)$ is covariantly constant.

Definition 2.3. Let $f_j^i(x, y)$ be an $f(3, 1)$ -Finsler structure of index K . A Finsler connection $F\Gamma$ is called an $f(3, 1)$ -Finsler connection, or compatible with f_j^i , (2.1), if:

$$(2.6) \quad f_{j|k}^i = 0, \quad f_j^i|_k = 0.$$

Proposition 2.5. With respect to a Finsler connection $F\Gamma$ compatible with and $f(3, 1)$ -Finsler structure $f_j^i(x, y)$, the tensor fields h_j^i and v_j^i covariantly constant:

$$(2.7) \quad h_{j|k}^i = 0, \quad v_{j|k}^i = 0, \quad h_j^i|_k = 0, \quad v_j^i|_k = 0.$$

Theorem 2.1. (i) The Obata tensor fields Q_{ij}^r and Q_{ij}^s are covariantly constant with respect to any $f(3, 1)$ -Finsler connection $F\Gamma$.

(ii) The Finsler tensor fields $Q_{sj}^r R_r^s{}_{kl}$, $Q_{sj}^r P_r^s{}_{kl}$, $Q_{sj}^r S_r^s{}_{kl}$ and their h - and v -covariant derivatives of every order vanish for every $F\Gamma$ with the property (2.6).

PROOF. The statement (i) results immediately from (2.4), (2.6) and (2.7). Applying the Ricci identities to f_j^i and taking into account (i) we get the property (ii).

Theorem 2.2. If on the differentiable manifold M there exists a Finsler connection $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$, then there exist $f(3, 1)$ -Finsler connections with respect to the $f(3, 1)$ -Finsler structure (2.1). One of these is:

$$(2.8) \quad \begin{cases} \hat{N}_j^i = \overset{\circ}{N}_j^i, & \hat{F}_{jk}^i = \overset{\circ}{F}_{jk}^i - \frac{1}{2} \{ f_h^i f_j^h \mathring{\mid}_k + v_j^i \mathring{\mid}_k - 3v_h^i v_j^h \mathring{\mid}_k \}, \\ \hat{C}_{jk}^i = \overset{\circ}{C}_{jk}^i - \frac{1}{2} \{ f_h^i f_j^h \mathring{\mid}_k + v_j^i \mathring{\mid}_k - 3v_h^i v_j^h \mathring{\mid}_k \}. \end{cases}$$

where $\mathring{\mid}$ and $\mathring{\mid}$ denote the h - and v -covariant derivatives with respect to $F\overset{\circ}{\Gamma}$.

PROOF. A straightforward calculation shows that the Finsler connection given by (2.8) satisfies equations (2.6).

We shall determine all $f(3, 1)$ -Finsler connections by a well known method based on Proposition 2.4. Let $F\overset{\circ}{\Gamma} = (\overset{\circ}{N}, \overset{\circ}{F}, \overset{\circ}{C})$ be a fixed Finsler connection on M . Then any Finsler connection $F\Gamma = (N, F, C)$ on M can be expressed in the form, [6]:

$$(2.9) \quad N_j^i = \overset{\circ}{N}_j^i - A_j^i, \quad F_{jk}^i = \overset{\circ}{F}_{jk}^i + \overset{\circ}{C}_{jr}^i A_k^r - B_{jk}^i, \quad C_{jk}^i = \overset{\circ}{C}_{jk}^i - D_{jk}^i,$$

where $A_j^i, B_{jk}^i, D_{jk}^i$ are arbitrary Finsler tensor fields.

We consider $F\overset{\circ}{\Gamma} = F\hat{\Gamma}$ in (2.9), where $F\hat{\Gamma} = (\hat{N}, \hat{F}, \hat{C})$ is given by (2.8). In order that $F\Gamma$ is an $f(3,1)$ -Finsler connection, that is, (2.6) holds for $F\Gamma$, it is necessary and sufficient that $A_j^i, B_{jk}^i, D_{jk}^i$ satisfy

$$B_{jk}^i + (f_s^i f_j^h - v_s^i v_j^h) B_{hk}^s = 0, \quad D_{jk}^i + (f_s^i f_j^h - v_s^i v_j^h) D_{hk}^s = 0,$$

which is equivalent to $Q \underset{2}{B} = 0, Q \underset{2}{D} = 0$. From Proposition 2.4, however, the last system has solutions in B_{jk}^i, D_{jk}^i for any Finsler tensor field $A_j^i = X_j^i$. Hence

Theorem 2.3. *Let $F\overset{\circ}{\Gamma}$ be a fixed Finsler connection. The set of all $f(3,1)$ -Finsler connections $F\Gamma$ with respect to the $f(3,1)$ -Finsler structure (2.1) is given by:*

$$(2.10) \quad \begin{cases} N_j^i = \overset{\circ}{N}_j^i - X_j^i \\ F_{jk}^i = \overset{\circ}{F}_{jk}^i + \overset{\circ}{C}_{jr}^i X_k^r - \frac{1}{2} \{ f_h^i (f_j^h \lrcorner_k + f_j^h \lrcorner_m X_k^m) + v_j^i \lrcorner_k + \\ \quad + v_j^i \lrcorner_m X_k^m - 3v_h^i (v_j^h \lrcorner_k + v_j^h \lrcorner_m X_k^m) \} + Q_{sj}^{ih} Y_{hk}^s, \\ C_{jk}^i = \overset{\circ}{C}_{jk}^i - \frac{1}{2} \{ f_h^i f_j^h \lrcorner_k + v_j^i \lrcorner_k - 3v_h^i v_j^h \lrcorner_k \} + Q_{sj}^{ih} Z_{hk}^s, \end{cases}$$

where $X_j^i, Y_{jk}^i, Z_{jk}^i$ are arbitrary Finsler tensor fields.

Remark 2.1. The $f(3,1)$ -Finsler connection $F\hat{\Gamma} = (\hat{N}, \hat{F}, \hat{C})$ given by (2.8) is obtained from (2.10) for $X_j^i = Y_{jk}^i = Z_{jk}^i = 0$.

Corollary 2.1. *If $F\overset{\circ}{\Gamma}$ is a fixed $f(3,1)$ -Finsler connection, then the set of all $f(3,1)$ -Finsler connections $F\Gamma$ is given by:*

$$(2.11) \quad \begin{aligned} N_j^i &= \overset{\circ}{N}_j^i - X_j^i, \\ F_{jk}^i &= \overset{\circ}{F}_{jk}^i + \overset{\circ}{C}_{jr}^i X_k^r + Q_{sj}^{ih} Y_{hk}^s, \\ C_{jk}^i &= \overset{\circ}{C}_{jk}^i - Q_{sj}^{ih} Z_{hk}^s, \end{aligned}$$

where $X_j^i, Y_{jk}^i, Z_{jk}^i$ are arbitrary Finsler tensor fields.

We denote by $F\Gamma(N)$ the Finsler connections having the same non-linear connection N .

Theorem 2.4. The set of all $f(3,1)$ -Finsler connections $FT(\overset{\circ}{N})$ is given by

$$(2.12) \quad \begin{cases} F_{jk}^i = \overset{\circ}{F}_{jk}^i - \frac{1}{2} \{ f_h^i f_j^h \overset{\circ}{\Gamma}_{jk}^i + v_j^i \overset{\circ}{\Gamma}_{jk}^i - 3v_h^i v_j^h \overset{\circ}{\Gamma}_{jk}^i \} + Q_{sj}^{ih} Y_{hk}^s, \\ C_{jk}^i = C_{jk}^i - \frac{1}{2} \{ f_h^i f_j^h \overset{\circ}{\Gamma}_{jk}^i + v_j^i \overset{\circ}{\Gamma}_{jk}^i - 3v_h^i v_j^h \overset{\circ}{\Gamma}_{jk}^i \} + Q_{sj}^{ih} Z_{hk}^s, \end{cases}$$

where $\overset{\circ}{F}\Gamma$ is a fixed Finsler connection, and Y_{jk}^i, Z_{jk}^i are arbitrary Finsler tensor fields.

3. The group of transformations of $f(3,1)$ -Finsler connections

Let us consider the transformations $FT(N) \rightarrow F\bar{\Gamma}(N)$ of $f(3,1)$ -Finsler connections that conserve the non-linear connection N . According to Theorem 2.4 they are given by:

$$(3.1) \quad \bar{N}_j^i = N_j^i, \quad \bar{F}_{jk}^i = F_{jk}^i + Q_{sj}^{ih} Y_{hk}^s, \quad \bar{C}_{jk}^i = C_{jk}^i + Q_{sj}^{ih} Z_{hk}^s.$$

Theorem 3.1. The set of all transformations (3.1) together with the mapping product is an Abelian group G_f isomorphic with the additive group of pairs of Finsler tensor fields $(Q_{sj}^{ih} Y_{hk}^s, Q_{sj}^{ih} Z_{hk}^s)$.

By a straightforward calculation we can prove:

Theorem 3.2. The following Finsler tensor fields are invariants by the action of the group G_f :

$$(3.2) \quad R_{jk}^i, \quad t_{jk}^i$$

$$(3.3) \quad \begin{aligned} \overset{1}{N}_{jk}^i &= Q_{pq}^{ri} Q_{jk}^{ps} T_{rs}^q, & \tilde{C}_{jk}^i &= Q_{pq}^{ri} Q_{jk}^{ps} C_{rs}^q, \\ \tilde{P}_{jk}^i &= Q_{pq}^{ri} Q_{jk}^{ps} P_{rs}^q, & \overset{2}{N}_{jk}^i &= Q_{pq}^{ri} Q_{jk}^{ps} S_{rs}^q, \end{aligned}$$

$$(3.4) \quad \begin{cases} \overset{1}{T}_{jk}^i = h_m^i T_{jk}^m - f_j^r f_k^s T_{rs}^i + (f_j^r T_{rk}^m + f_k^s T_{js}^m) f_m^i, \\ \overset{1}{R}_{jk}^i = h_m^i R_{jk}^m - f_j^r f_k^s R_{rs}^i + (f_j^r R_{rk}^m + f_k^s R_{js}^m) f_m^i, \\ \overset{1}{C}_{jk}^i = h_m^i C_{jk}^m - f_j^r f_k^s C_{rs}^i + (f_j^r C_{rk}^m + f_k^s C_{js}^m) f_m^i, \\ \overset{1}{P}_{jk}^i = h_m^i P_{jk}^m - f_j^r f_k^s P_{rs}^i + (f_j^r P_{rk}^m + f_k^s P_{js}^m) f_m^i, \\ \overset{1}{S}_{jk}^i = h_m^i S_{jk}^m - f_j^r f_k^s S_{rs}^i + (f_j^r S_{rk}^m + f_k^s S_{js}^m) f_m^i, \end{cases}$$

$$(3.5) \quad \begin{cases} {}^2R_{jk}^i = h_m^i R_{jk}^m - f_j^r f_k^s R_{rs}^i - (f_j^r R_{rk}^m + f_k^s R_{js}^m) f_m^i, \\ {}^2\tilde{C}_{jk}^i = h_m^i C_{jk}^m - f_j^r f_k^s C_{rs}^i + (f_j^r C_{rk}^m - f_k^s C_{js}^m) f_m^i, \\ {}^2P_{jk}^i = h_m^i P_{jk}^m + f_j^r f_k^s P_{rs}^i - (f_j^r P_{rk}^m - f_k^s P_{js}^m) f_m^i, \end{cases}$$

$$(3.6) \quad \begin{cases} {}^3T_{jk}^i = h_m^i T_{jk}^m - (f_j^r P_{kr}^m - f_k^s P_{jr}^m) f_m^i \\ {}^3R_{jk}^i = h_m^i R_{jk}^m - f_j^r f_k^s S_{rs}^i - (f_j^r C_{kr}^m - f_k^s C_{js}^m) f_m^i, \\ {}^3\tilde{C}_{jk}^i = h_m^i C_{jk}^m + f_j^r f_k^s C_{sr}^i + (f_j^r S_{rk}^m + f_k^s R_{js}^m) f_m^i, \\ {}^3P_{jk}^i = h_m^i P_{jk}^m + f_j^r f_k^s P_{sr}^i + f_k^s T_{js}^m f_m^i \\ {}^3S_{jk}^i = h_m^i S_{jk}^m - f_j^r f_k^s R_{rs}^i + (f_j^r C_{rk}^m - f_k^s C_{sj}^m) f_m^i, \\ \tilde{T}_{jk}^i = f_j^r f_k^s T_{rs}^i - (f_j^r P_{rk}^m - f_k^s P_{sj}^m) f_m^i. \end{cases}$$

Theorem 3.3. *The Finsler tensor fields $\overset{1}{N}_{jk}^i$, $\overset{2}{N}_{jk}^i$, $\overset{1}{T}_{jk}^i$, $\overset{1}{S}_{jk}^i$ vanish if and only if there exists on M a h - and v -semi-symmetric $f(3,1)$ -Finsler connection $FT(N)$.*

Remark 3.1. If the $f(3,1)$ -Finsler structure is of index null, that is, if it is an almost complex Finsler structure ($h_j^i = \delta_j^i$, $v_j^i = 0$), then we have $\overset{1}{N} = \overset{1}{T} = \overset{*}{N}$, $\overset{1}{R} = \overset{*}{R}$, $\tilde{C} = \overset{1}{C} = \overset{*}{C}$, $\tilde{P} = \overset{1}{P} = \overset{*}{P}$, $\overset{2}{N} = \overset{1}{S} = \overset{**}{N}$, $\overset{2}{R} = \overset{**}{R}$, $\overset{2}{C} = \overset{**}{C}$, $\overset{2}{P} = \overset{**}{P}$, and (3.6) can be constructed with $\overset{3}{R}_{jk}^i = \overset{**}{S}_{jk}^i$, and $f_m^i f_j^r f_k^s \tilde{T}_{jk}^m = \overset{**}{T}_{jk}^i$, only (cf. with R. MIRON [7], Theorem 3.2, where there are $\overset{*}{N}, \overset{*}{R}, \dots, \overset{*}{T}$).

4. $\tilde{F}(3,1)$ -structures on the tangent bundle

Let N be a fixed non-linear connection on TM .

An $\tilde{F}(3,1)$ -structure of index k' on TM is given by a tensor field $\tilde{F} \in \tau_1^1(TM)$ with property:

$$(4.1) \quad \begin{aligned} \tilde{F}^3 + \tilde{F} &= 0 \quad \text{rank } \|\tilde{F}(x, y)\| = 2n - k' = 2p', \\ 0 &\leq k' < 2n, \forall (x, y) \in TM \end{aligned}$$

In the adapted basis $X_A = \{\delta_i, \dot{\partial}_i\}$, $A = \overline{1, 2n}$, $i = \overline{1, n}$, $\tilde{F}$ can be represented by:

$$(4.2) \quad \begin{aligned} \tilde{F} = & \overset{1}{F}_j^i \delta_i \otimes dx^j + \overset{2}{F}_j^i \delta_i \otimes \delta y^j + \\ & + \overset{3}{F}_j^i \dot{\partial}_i \otimes dx^j + \overset{4}{F}_j^i \dot{\partial}_i \otimes \delta y^j, \end{aligned}$$

where $\overset{\alpha}{F}_j^i$ ($\alpha = 1, 2, 3, 4$) are Finsler tensor on M . Then we have

$$(4.3) \quad \tilde{F}(\delta_j) = \overset{1}{F}_j^i \delta_i + \overset{3}{F}_j^i \dot{\partial}_i, \quad \tilde{F}(\dot{\partial}_j) = \overset{2}{F}_j^i \delta_i + \overset{4}{F}_j^i \dot{\partial}_i,$$

and the condition (4.1) is equivalent with

$$(4.4) \quad \left\{ \begin{aligned} & (\overset{1}{F}_h^i \overset{1}{F}_s^h + \overset{2}{F}_h^i \overset{3}{F}_s^h) \overset{1}{F}_j^s + (\overset{1}{F}_h^i \overset{2}{F}_s^h + \overset{2}{F}_h^i \overset{4}{F}_s^h) \overset{3}{F}_j^s + \overset{1}{F}_j^i = 0, \\ & (\overset{1}{F}_h^i \overset{1}{F}_s^h + \overset{2}{F}_h^i \overset{3}{F}_s^h) \overset{2}{F}_j^s + (\overset{1}{F}_h^i \overset{2}{F}_s^h + \overset{2}{F}_h^i \overset{4}{F}_s^h) \overset{4}{F}_j^s + \overset{2}{F}_j^i = 0, \\ & (\overset{3}{F}_h^i \overset{1}{F}_s^h + \overset{4}{F}_h^i \overset{3}{F}_s^h) \overset{1}{F}_j^s + (\overset{3}{F}_h^i \overset{2}{F}_s^h + \overset{4}{F}_h^i \overset{4}{F}_s^h) \overset{3}{F}_j^s + \overset{3}{F}_j^i = 0, \\ & (\overset{3}{F}_h^i \overset{1}{F}_s^h + \overset{4}{F}_h^i \overset{3}{F}_s^h) \overset{2}{F}_j^s + (\overset{3}{F}_h^i \overset{2}{F}_s^h + \overset{4}{F}_h^i \overset{4}{F}_s^h) \overset{4}{F}_j^s + \overset{4}{F}_j^i = 0, \end{aligned} \right.$$

Also, we suppose that the components of $\tilde{F}$ fulfil the conditions

$$(4.5) \quad H = -\tilde{F}^2, \quad V = \tilde{F}^2 + I,$$

so as to be orthogonal and supplementary projectors.

The Nijenhuis tensor of $\tilde{F}$ is given by

$$(4.6) \quad \begin{aligned} \tilde{N}(X, Y) = & H[X, Y] + \tilde{F}[\tilde{F}X, Y] + \tilde{F}[X, \tilde{F}Y] - \\ & - [\tilde{F}X, \tilde{F}Y], \quad \forall X, Y \in \Xi(TM). \end{aligned}$$

The integrability condition of the $\tilde{F}(3, 1)$ -structure $\tilde{F}$ is $\tilde{N}(X, Y) = 0 \quad \forall X, Y \in \Xi(TM)$, [4], [9]. It is sufficient to calculate $\tilde{N}(\delta_j, \delta_k)$, $\tilde{N}(\delta_j, \dot{\partial}_k)$ and $\tilde{N}(\dot{\partial}_j, \dot{\partial}_k)$, and we can determine $\tilde{N}(X, Y)$.

If $f_j^i(x, y)$ is an $f(3, 1)$ -Finsler structure of index K on M , then on TM , in the presence of a non-linear connection, we have some important special cases:

$$(4.7) \quad \left\{ \begin{aligned} & \overset{1}{\tilde{F}} = f_j^i \delta_i \otimes dx^j + f_j^i \dot{\partial}_i \otimes \delta y^j, \\ & \overset{2}{\tilde{F}} = f_j^i \delta_i \otimes dx^j - f_j^i \dot{\partial}_i \otimes \delta y^j, \\ & \overset{3}{\tilde{F}} = f_j^i \delta_i \otimes \delta y^j + f_j^i \dot{\partial}_i \otimes dx^j. \end{aligned} \right.$$

The tensor fields $\tilde{F}^\alpha$ ($\alpha = 1, 2, 3$) given by (4.6) are $\tilde{F}(3, 1)$ -structures of special type on TM . Indeed, conditions (2.2) and (2.3) being fulfilled for f_j^i , we get

$$(4.8) \quad \left\{ \begin{array}{l} \tilde{F}^3 + \tilde{F}^\alpha = 0 \quad \text{rank } \|\tilde{F}^\alpha(x, y)\|_{\mathcal{H}TM} = \\ \quad \quad \quad = \text{rank } \|\tilde{F}^\alpha(x, y)\|_{\mathcal{V}TM} = n - k \\ H = -\tilde{F}^2 = h_j^i \delta_j \otimes dx^j + h_j^i \dot{\partial}_i \otimes \delta y^j \\ V = \tilde{F}^2 + I = v_j^i \delta_i \otimes dx^j + v_j^i \dot{\partial}_i \otimes \delta y^j, \quad \forall \alpha = 1, 2, 3, \end{array} \right.$$

where $h_j^i = -f_h^i f_j^h$, $v_j^i = f_h^i f_j^h + \delta_j^i$.

To the formulae (4.7) we give $\tilde{F}(3, 1)$ -structures on TM by the lift of an $f(3, 1)$ -Finsler structure from M to the total space TM of the tangent bundle $T(M)$.

5. The integrability of $f(3, 1)$ -Finsler structures

Let N be a non-linear connection of $T(M)$. Then an $f(3, 1)$ -Finsler structure on the base manifold M can be lifted to an $\tilde{F}(3, 1)$ -structure on $T(M)$ in three manners (4.7). The values of the Finsler components of $\tilde{F}$ from (4.2) are given in the following table:

$\tilde{F}$	$\tilde{F}_j^1$	$\tilde{F}_j^2$	$\tilde{F}_j^3$	$\tilde{F}_j^4$
$\tilde{F}^1$	f_j^i	0	0	f_j^i
$\tilde{F}^2$	f_j^i	0	0	$-f_j^i$
$\tilde{F}^3$	0	f_j^i	f_j^i	0

We remark the following relations

$$\begin{aligned} \tilde{F}^1(\delta_j) &= f_j^i \delta_i, & \tilde{F}^1(\dot{\partial}_j) &= f_j^i \dot{\partial}_i, \\ \tilde{F}^2(\delta_j) &= f_j^i \delta_i, & \tilde{F}^2(\dot{\partial}_j) &= -f_j^i \dot{\partial}_i, \\ \tilde{F}^3(\delta_j) &= f_j^i \dot{\partial}_i, & \tilde{F}^3(\dot{\partial}_j) &= f_j^i \delta_i. \end{aligned}$$

Definition 5.1. An $f(3,1)$ -Finsler structure of index K on a differentiable manifold M is called integrable of type *I*, *II* or *III* with respect to the non-linear connection N , if the corresponding lifted $\tilde{F}^1(3,1)$ -, $\tilde{F}^2(3,1)$ -, or $\tilde{F}^3(3,1)$ -structure is integrable.

We characterize these cases of integrability using only the invariants of the group G_f .

Theorem 5.1. The $f(3,1)$ -Finsler structure, (2.1) is integrable of type *I*, *II* or *III* if and only if the invariants of the group G_f have the values given in the following table

Type of integrability	Characterization by invariants
<i>I</i>	$\tilde{T}_{jk}^i = 0; \tilde{R}_{jk}^i = 0; \tilde{C}_{jk}^i = 0; \tilde{P}_{jk}^i = 0, \tilde{S}_{jk}^i = 0$
<i>II</i>	$\tilde{T}_{jk}^i = 0; \tilde{R}_{jk}^i = 0; \tilde{C}_{jk}^i = 0; \tilde{P}_{jk}^i = 0, \tilde{S}_{jk}^i = 0$
<i>III</i>	$\tilde{T}_{jk}^i = 0; \tilde{R}_{jk}^i = 0; \tilde{C}_{jk}^i = 0; \tilde{P}_{jk}^i = 0,$ $\tilde{S}_{jk}^i = 0; \tilde{T}_{jk}^i = 0$

PROOF. The $f(3,1)$ -Finsler structure is integrable of type *I* if and only if $\tilde{N}(X, Y) = 0$ for $\tilde{F}^1$. But $\tilde{N}(X, Y) = 0, \forall X, Y \in \Xi(TM)$ is equivalent to

$$\tilde{N}(\delta_j, \delta_k) = 0, \quad \tilde{N}(\delta_j, \dot{\partial}_k) = 0, \quad \tilde{N}(\dot{\partial}_j, \dot{\partial}_k) = 0,$$

which are equivalent to $\tilde{T}_{jk}^i = 0, \tilde{R}_{jk}^i = 0, \tilde{C}_{jk}^i = 0, \tilde{P}_{jk}^i = 0, \tilde{S}_{jk}^i = 0$. The proofs of *II* and *III* follow the same pattern.

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