

Nonlinear connection in cotangent bundle

By GH. ATANASIU (Braşov) and F.C. KLEPP (Timişoara)

*Dedicated to Professor Lajos Tamásy on the occasion
of his 65th birthday*

1. Introduction

In the present paper we give a dual theory corresponding to the theory of nonlinear connections in Lagrange spaces, given by the first author in [1].

We study the possibility of determination of a nonlinear connection N on the cotangent bundle $T^*(M) = (T^*M, \pi^*, M)$ of a differentiable manifold M , starting from a linear connection $\bar{\nabla}^v$ on the vertical subbundle VT^*M of T^*M . We also study the possibility of the determination of a distinguished connection $F^*\Gamma = (N_{jk}, F_j^i, C_i^{jk})$ on T^*M , whose restriction to VT^*M coincides with $\bar{\nabla}^v$.

The terminology and notations follow those in MIRON's papers [4], [5], [6], with some modifications given in ATANASIU-HASHIGUCHI's paper [2]. For convenience sake, in §2 we shall sketch the material from MIRON's theory necessary for our discussions.

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2. Distinguished geometrical objects on the cotangent bundle

Let M be an n -dimensional C^∞ -manifold and $\pi^* : T^*M \rightarrow M$ its cotangent bundle. Since a point of T^*M is a covector (x, p) at a point x of the base manifold M , a coordinate system $x = (x^i)$ in M induces a canonical coordinate system $(x, p) = (x^i, p_i)$ in T^*M by $p = p_i(dx^i)$.

A transformation of the canonical coordinate system in T^*M is given by

$$(2.1) \quad \tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n), \quad \tilde{p}_i = \frac{\partial x^r}{\partial \tilde{x}^i} p_r; \quad \det \left(\frac{\partial \tilde{x}^i}{\partial x^r} \right) \neq 0$$

The kernel of the differential of the mapping $\pi^* : T^*M \rightarrow M$ gives us the vertical subbundle VT^*M of the tangent bundle TT^*M . If $V_{(x,p)}$ is the fiber of VT^*M at the point $(x,p) \in T^*M$, then we have the vertical distribution $V : (x,p) \in T^*M \rightarrow V_{(x,p)} \subset T_{(x,p)}T^*M$.

A *nonlinear connection* N on T^*M is a distribution of class C^∞ given by $(x,p) \in T^*M \rightarrow N_{(x,p)} \subset T_{(x,p)}T^*M$ such that $T_{(x,p)}T^*M = N_{(x,p)} \oplus V_{(x,p)}$. N is spanned by

$$(2.2) \quad \delta_i = \partial_i + N_{ri}(x,p)\dot{\partial}^r,$$

where $\partial_i = \partial/\partial x^i$, $\dot{\partial} = \partial/\partial p_i$. $\{\delta_i, \dot{\partial}_i\}$ is a local basis adapted to the distributions N and V , and its dual basis is $\{dx^i, \delta p_i\}$, where

$$(2.3) \quad \delta p_i = dp_i - N_{ir}dx^r.$$

The functions $N_{jk}(x,p)$ are called the *coefficients* of the nonlinear connection N and these are uniquely determined.

With respect to (2.1), we have

$$(2.4) \quad \begin{aligned} \tilde{\delta}_i &= \frac{\partial x^r}{\partial \tilde{x}^i} \delta_r, & \tilde{\dot{\partial}}^i &= \frac{\partial \tilde{x}^i}{\partial x^r} \dot{\partial}^r, \\ d\tilde{x}^i &= \frac{\partial \tilde{x}^i}{\partial x^r} dx^r, & \delta \tilde{p}_i &= \frac{\partial x^r}{\partial \tilde{x}^i} \delta p_r, \end{aligned}$$

and the transformation formula of N_{jk} with respect to (2.1) is

$$(2.5) \quad \tilde{N}_{jk}(\tilde{x}, \tilde{p}) = \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} N_{st}(x,p) + p_r \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k}.$$

A *distinguished tensor field*, a *d-tensor field* for short, of type (r,s) on M is defined by its components, satisfying the classical law of transformation with respect to (2.1).

A *d-connection* on M is a triad $F^*\Gamma = (N, F, C)$, where $N = (N_{jk})$ is a nonlinear connection, $F = (F_j^{ik})$ is a special *d-object* and $C = (C_i^{jk})$ is a *d-tensor field* of type (2.1). The transformation formulal of N_{jk} , F_j^{ik} , C_i^{jk} with respect to (2.1) are (2.5) and

$$(2.6) \quad \tilde{F}_j^{ik} = \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} F_s^{rt} + \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k},$$

$$(2.7) \quad \tilde{C}_i^{jk} = \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial \tilde{x}^j}{\partial x^s} \frac{\partial \tilde{x}^k}{\partial x^t} C_r^{st}.$$

Given a d -connection, the h - and v -covariant derivatives are defined for a d -tensor field, e.g. for K_j^i , by

$$(2.8) \quad K_{j|k}^i = \delta_k K_j^i + K_j^r F_r^i{}_k - K_r^i F_j^r{}_k, \quad K_j^i|{}^k = \dot{\partial}^k K_j^i + K_j^r C_r^{jk} - K_r^i C_j^{rk}.$$

For example the, h - and v -covariant derivatives of the d -covector p_i are:

$$(2.9) \quad p_{i|h} = N_{ih} - F_i^o{}_h, \quad p_i|{}^h = \delta_i^h - C_i^{oh},$$

where the index "o" means the contraction with p_i .

We denote

$$(2.10) \quad D_{ih} = N_{ih} - F_i^o{}_h, \quad d_i^h = \delta_i^h - C_i^{oh}.$$

The d -tensor field D_{ih} is called the h -deflection tensor field and d_i^h is called the v -deflection tensor field of the d -connection $F^*\Gamma$ on T^*M .

3. Nonlinear connections and d -connections on T^*M derived from linear connections given only on the vertical subbundle VT^*M

Let us denote by $\overset{*}{\Gamma}_j^i{}_k$, C_i^{jk} the coefficients of the linear connection $\overset{*}{\nabla}^v$ obtained by restricting the d -connection $F^*\Gamma$ on the vertical subbundle VT^*M :

$$(3.1) \quad \overset{*}{\nabla}_{\partial_k}^v \dot{\partial}^i = -\overset{*}{\Gamma}_j^i{}_k \dot{\partial}^j, \quad \overset{*}{\nabla}_{\dot{\partial}^k}^v \dot{\partial}^j = -C_i^{jk} \dot{\partial}^i.$$

Effecting a change of the canonical coordinates $(x, p) \rightarrow (\tilde{x}, \tilde{p})$ on T^*M we can deduce by a simple calculation the transformation law of $\overset{*}{\Gamma}_j^i{}_k$

$$(3.2) \quad \tilde{\overset{*}{\Gamma}}_j^i{}_k = \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} \overset{*}{\Gamma}_s^i{}_t + \left(\delta_j^r \frac{\partial \tilde{x}^i}{\partial x^m} - \tilde{C}_j^{ir} p_m \right) \frac{\partial^2 x^m}{\partial \tilde{x}^r \partial \tilde{x}^k},$$

and C_i^{jk} has the transformation law given by (2.7).

There arises the following problem: given a linear connection

$\overset{*}{\nabla}^v \left(\overset{*}{\Gamma}_j^i{}_k, C_i^{jk} \right)$ on the vertical subbundle VT^*M , under which conditions

does $\overset{*}{\nabla}^v$ determine

(i) a nonlinear connection on T^*M

(ii) a d -connection on T^*M whose restriction to VT^*M coincides with $\overset{*}{\nabla}^v$.

To solve this problem we first give the following definition.

Definition 3.1. A linear connection ∇^* on the vertical bundle is called *strongly regulated* if the v -deflection tensor field $d_i^k = \delta_i^k - C_i^{ok}$ is non-degenerate, i.e. $\|d_i^k\| = n$.

We show that for a strongly regulated linear connection ∇^* the above problem has a natural solution. – In fact, we multiply (3.2) with $\tilde{p}_i$, and we get

$$(3.3) \quad \tilde{\Gamma}_j^o{}_k = \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} \tilde{\Gamma}_s^o{}_t + \tilde{d}_j^r p_m \frac{\partial^2 x^m}{\partial \tilde{x}^r \partial \tilde{x}^k}.$$

Since ∇^* is strongly regulated, the inverse of the matrix (d_j^r) exists, and it will be denoted by $(d_j^r)^{-1} = (b^r{}_j)$. Then by multiplying (3.3) with $\tilde{b}^j{}_i$ we obtain

$$(3.4) \quad \tilde{b}^j{}_i \tilde{\Gamma}_j^o{}_k = \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} b^r{}_t \tilde{\Gamma}_t^o{}_s + p_m \frac{\partial^2 x^m}{\partial \tilde{x}^i \partial \tilde{x}^k}.$$

We define

$$(3.5) \quad N_{ik} = b^r{}_i \tilde{\Gamma}_r^o{}_k,$$

and from (3.4) we get

$$(3.6) \quad \tilde{N}_{ik} = \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} N_{rs} + p_m \frac{\partial^2 x^m}{\partial \tilde{x}^i \partial \tilde{x}^k}$$

that is, the transformation law of the nonlinear connection on T^*M .

In this way have obtained

Theorem 3.1. Let be given a strongly regulated linear connection ∇^* on the vertical subbundle VT^*M of the tangent bundle $\pi^* : T^*M \rightarrow M$, with coefficients $(\tilde{\Gamma}_j^i{}_k, C_i^{jk})$, then:

- (a) there exists a global nonlinear connection N defined on T^*M , and determined only by ∇^* . The coefficients N_{jk} of N are given by the formulal (3.5), where $(b^i{}_j) = (d_j^i)^{-1}$;
- (b) the triad $(N_{jk}, F_j^i{}_k, C_i^{jk})$ with

$$(3.7) \quad F_j^i{}_k = \tilde{\Gamma}_j^i{}_k + C_j^{ir} N_{kr}$$

is a d -connection, which depends only on the vertical connection ∇^* and whose restriction at VT^*M is ∇^* .

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GHEORGHE ATANASIU
CATEDRA DE MATEMATICĂ
UNIVERSITATEA DIN BRAŞOV
2200-BRAŞOV, ROMANIA

FRANCISC C. KLEPP
CATEDRA DE MATEMATICI
INST. POLITEHNIC "TRAIAN VUIA"
1900-TIMIŞOARA, ROMANIA

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