Publ. Math. Debrecen
39 / 1–2 (1991), 149–153

Multiplicative functions satisfying a congruence property. III

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Let $\mathcal{M}$ be the set of integer-valued multiplicative functions. For a fixed positive integer M let $K_M \subseteq \mathcal{M}$ denote the set of those functions $f(n)$ for which

$$\Delta_M f(n+p) \equiv \Delta_M f(n) \pmod{p} \tag{1}$$

holds for every positive integer n and for every prime p, where

$$\Delta_M f(n) = f(n+M) - f(n).$$

From a result of [1] it follows that if $f \in K_1$, then $f(n)$ is a power of n with non-negative integer exponent. The aim of this note is to extend this result for K_M. We shall prove the following.

Theorem. *Let M be a positive integer and let $f \in K_M$. Then for every positive integer n either*

$$f(n) = n^a \quad \text{or} \quad f(n+M) = f(n),$$

where a is a positive integer.

We need two lemmas for the proof of our theorem.

Lemma 1. *Let $f \in K_M$. Then for each prime p with $p^e \| M$ $(e \geq 0)$ and $f(p^e) \neq 0$, we have*

$$f(p^{e+k}) = \left(\frac{f(p^{e+1})}{f(p^e)}\right)^k f(p^e) \quad (k = 1, 2, \dots). \tag{2}$$

* Research partially supported by Hungarian National Foundation for Scientific Research grant No. 907.

PROOF. For a positive integer m let m^* denote the product of all distinct prime factors of m. First we note that by the definition of K_M it follows that if $f \in K_M$, then

$$\Delta_{tM} f(n+m) \equiv \Delta_{tM} f(n) \pmod{m^*} \tag{3}$$

for every positive integer n, m, t. Thus

$$\begin{aligned} f(n+hmM) - f(n) &= \sum_{i=0}^{h-1} \{f(n+(i+1)mM) - f(n+imM)\} = \\ &= \sum_{i=0}^{h-1} \Delta_{mM} f(n+imM) \equiv \sum_{i=0}^{h-1} \Delta_{mM} f(n) = \\ &= h\Delta_{mM} f(n) \pmod{(mM)^*} \end{aligned} \tag{4}$$

and so, replacing h by hm^*,

$$f(n+hmm^*M) \equiv f(n) \pmod{m^*} \tag{5}$$

holds for every positive integer n, m and h.

We prove that for an integer $m \geq 2$ and for a positive integer n with $(n, M) = 1$

$$m^* | f(n) \quad \text{implies} \, (n, m^*) > 1. \tag{6}$$

Assume indirectly that for an integer $m \geq 2$ and for a positive integer n with $(n, M) = 1$

$$m^* | f(n) \quad \text{and} \quad (n, m^*) = 1.$$

Then by Dirichlet's theorem there exist positive integers x, y such that

$$(n, x) = 1 \text{ and } nx = 1 + mm^*My. \tag{7}$$

Applying (5), by using (7) we have

$$0 \equiv f(n)f(x) = f(nx) = f(1+mm^*My) \equiv f(1) = 1 \pmod{m^*},$$

which is a contradiction, since $m \geq 2$. Thus (6) is proved.

Let p be a prime number with $p^e \| M \, (e \geq o)$ and $f(p^e) \neq 0$. We shall prove that for each positive integer $k > 0$

$$f(p^{e+k}) = \frac{f(p^{e+1})}{f(p^e)} f(p^{e+k-1}). \tag{8}$$

Let $q(>p)$ be an abritrary prime. For each prime q there exist positive integers $u=u(q)$, $v=v(q)$ such that

$$(u,pM)=1 \quad \text{and} \quad p^{e+k-1}u=p^e+q^2Mv. \tag{9}$$

Since $(u,qM)=1$, using (6) we have

$$f(u)\not\equiv 0 \pmod{q}. \tag{10}$$

Applying (5), by using (9) and (10) we get

$$\begin{aligned} f(p^{e+k}u)f(p^e) &\equiv f(p^{e+1})f(p^e) \\ &\equiv f(p^{e+1})f(p^{e+k-1}u) \pmod{q} \end{aligned}$$

and so by the multiplicativity of f

$$f(p^{e+k})\equiv \frac{f(p^{e+1})}{f(p^e)}f(p^{e+k-1}) \pmod{q},$$

which proves (8), since $q(>p)$ is an abritrary prime, and from (8) the lemma follows.

Lemma 2. *Let $f\in K_M$. Then there is a non-negative integer a such that*

$$f(nM)=n^a f(M) \tag{11}$$

for every positive integer n.

PROOF. First we prove that

$$f(nM+mM)\equiv f(nM) \pmod{m^*} \tag{12}$$

holds for every positive integer n,m.

By (3) and (4), applying (3) with $m=mM$, $n=M$, $t=n-1$, we can derive

$$\begin{aligned} f(nM+hmM)-f(nM) &\equiv h\Delta_{mM}f(nM) \\ &\equiv h\Delta_{mM}f(M) \pmod{m^*} \end{aligned} \tag{13}$$

for every positive integer n,m,h. Let $m=p$ be a prime for which $p^e\|M$ $(e\geq o)$ and $M=p^eM'$. By using (13) we get

$$f(pM+p^2M)-f(pM)=f(p^{e+1})\{f(M'+pM')-f(M')\}\equiv 0 \pmod{p}.$$

If

$$f(M' + pM') - f(M') \equiv 0 \pmod{p} \quad \text{or} \quad f(p^e) \equiv 0 \pmod{p},$$

then by (13)

$$f(nM + hpM) - f(nM) \equiv h\Delta_{pM} f(M)$$
$$= hf(p^e) \{ f(M' + pM') - f(M') \} \equiv 0 \pmod{p},$$

from which (12) follows. If

$$f(p^{e+1}) \equiv 0 \pmod{p} \quad \text{and} \quad f(p^e) \not\equiv 0 \pmod{p},$$

then by using Lemma 1, we have

$$f(p^{e+k}) \equiv 0 \pmod{p}$$

for $k = 1, 2, \ldots$, and so (13) implies that

$$f(M + pM) - f(M) \equiv f(pM + pM) - f(pM) \equiv 0 \pmod{p}.$$

This with (13) implies that

$$f(nM + hpM) - f(nM) \equiv 0 \pmod{p},$$

for any $n > 0$, from which (12) also follows. Thus (12) is proved.

Let q be a fixed prime, for which $(q, M) = 1$. Then, using (6) (see the proof of Lemma 1),

$$f(q) = \pm q^a, \tag{14}$$

where $a = a(q)$ is a non-negative integer. Let n be a positive integer.

Applying (12) and using Lemma 1 we have

$$f(nq^{2s}M) \equiv f(M) \pmod{(nq^{2s} - 1)^*}$$

and so

$$f(q^{2s})f(nM) \equiv f(M) \pmod{(nq^{2s} - 1)^*}. \tag{15}$$

By (14) and (15) it follows that

$$f(nM) = n^a f(M),$$

because $f(q^k) = f(q)^k$ $(k = 1, 2, \ldots)$ by Lemma 1, and

$$(nq^{2s} - 1)^* \to \infty \quad \text{as} \quad s \to \infty.$$

Thus Lemma 2 is proved.

PROOF. of the theorem. First we assume that $f(M) \neq 0$. Then, by Lemma 2, we have

$$f(n) = n^a \quad \text{if} \quad (n, M) = 1. \tag{16}$$

Let n be an arbitrary positive integer. Then there exist infinitely many positive integers m, for which

$$(n + m, M) = 1 \quad \text{and} \quad m^* \to \infty. \tag{17}$$

Since $f \in K_M$, for each integer m satisfying (17), by using (3) and (16), we have

$$\begin{aligned} f(n + tM) - f(n) &\equiv f(n + m + tM) - f(n + m) = \\ &= (n + m + tM)^a - (n + m)^a \equiv \\ &\equiv (n + tM)^a - n^a \pmod{m^*} \end{aligned}$$

and so

$$f(n + tM) - f(n) = (n + tM)^a - n^a. \tag{18}$$

Applying (18) in the case $t = n^2$, using (16) we get

$$f(n)\{(1 + nM)^a - 1\} = n^a\{(1 + nM)^a - 1\}.$$

If $a > 0$, then

$$f(n) = n^a \tag{19}$$

for every positive integer n. If $a = 0$, then by (18) it follows that

$$f(n + M) = f(n) \tag{20}$$

for every positive integer n.

Now assume that $f(M) = 0$. In this case, using Lemma 2,

$$f(nM) = n^a f(M) = 0 \tag{21}$$

for every positive integer n. Since $f \in K_M$, by using (21) and (3) we have

$$f(n + M) - f(n) \equiv f(ntM + M) - f(ntM) = 0 \pmod{(Mt - 1)^*},$$

which implies (20), because

$$(Mt - 1)^* \to \infty \quad \text{as} \quad t \to \infty.$$

This completes the proof of our theorem.

References

[1] B. M. Phong, Multiplicative functions satisfying a congruence property II, *Ann. Univ. Sci. Budapest, Sec. Math.* **33** (1990).

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(Received December 9, 1988)